MATHEMATICAL PROOF OF THE MANDEL-CRYER EFFECT IN POROELASTICITY*

C.J. VAN DUIJN† AND ANDRO MIKELIĆ‡

Abstract. We consider Mandel's problem from poroelasticity, which describes the behaviour of a water saturated porous sample being sandwiched between two rigid plates. It was observed, both computationally and experimentally, that the pore pressure in the center of the sample increases for some time and decreases later. This is known as the Mandel-Cryer effect.

It is the purpose of this paper to provide a rigorous mathematical setting for Mandel's problem and for the corresponding Mandel-Cryer effect. We first formulate non-standard linear parabolic problems for the volume strain and the fluid pressure. These problems admit "explicit" solutions in terms of Fourier series. Introducing the abstract variational parabolic formulation with appropriate spaces, the Fourier series are shown to converge strongly.

The main result is the mathematical proof of the Mandel-Cryer effect. Here we use the Laplace Transform applied to the pressure equation. We write the transformed pressure in such a way, that a Tauberian type of result applies to its time derivative. From this the Mandel-Cryer effect is immediate.

Key words. poroelasticity, Mandel's problem, Mandel-Cryer effect

AMS subject classifications. 35Q74, 74H10, 76S99

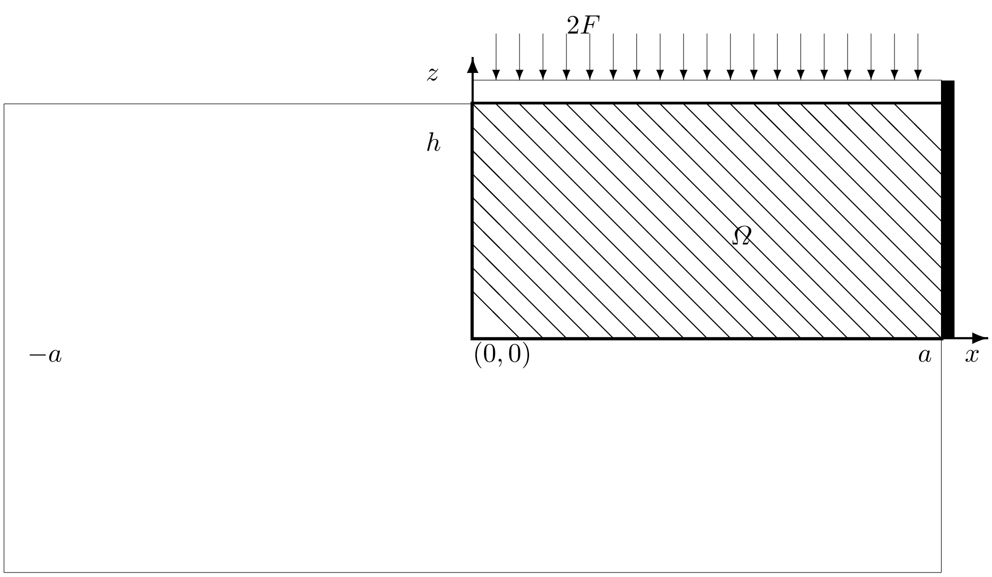

Fig. 1. *Geometrical setup of Mandel's problem. Because of symmetry we consider only the right-upper quarter of the domain.*

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1. Introduction.

In poroelasticity one describes, in essence, the behaviour of a deformable porous skeleton filled with a fluid. In it's simplest setting, the skeleton behaves linearly elastic and the fluid and grains are incompressible. Pioneering references are Biot[4], Terzaghi[21] and more recently Coussy[6] and Verruijt[23]. The equations describing poroelastic behaviour involve the skeleton displacement and the fluid pressure. They are coupled, time dependent and often multi-dimensional. Hence it is not straightforward to solve them numerically, let alone analytically. However, there is a well-known problem, called Mandel's problem (Mandel [15]), which allows for an explicit solution. The paper is devoted to Mandel's problem and the corresponding behaviour of the fluid pressure.

In Mandel's problem one considers an infinitely long rock sample having a rectangular cross section as shown on Figure 1. The sample is fully water saturated and sandwiched at top and bottom by two rigid, frictionless plates that act as no-flow boundaries for the fluid. Along the plates a uniform load of $2F$ $[Pa]$, where $[*]$ denotes the unit, is applied at $t = 0+$. This load is maintained at its constant value for all $t > 0$. The lateral boundaries $\{x = \pm a\}$ are drained and stress free. The sample is forced to be in plain strain conditions by preventing any deformation in the perpendicular direction. By symmetry, we may restrict our considerations to the upper right quadrant

$$\Omega = \{(x, z) : \ 0 < x < a, \quad 0 < z < h\}. \tag{1.1}$$

When the physical parameters of the model are constant, Mandel's problem admits an explicit solution that expresses the fluid pressure and the volume strain, corresponding to the effective solid skeleton displacement, in terms of infinite series. For this reason it is used as a benchmark for testing the validity of numerical simulations (Phillips [17], Phillips & Wheeler [18]).

The explicit series solution attracted quite some attention in the engineering literature, see for instance Abousleiman et al [2], Coussy [6] or Verruijt [22]. These authors observed from the pressure expansion, that the pressure in the center of the sample, at $\{x = 0\}$, shows non-monotone behavior: for small $t > 0$ the pressure rises above its value at $t = 0+$ and decreases for large t, see Figure 2 where a computational result is shown. This non-monotone pressure behavior is known as the Mandel-Cryer effect, since Cryer [7] observed similar behaviour for the pressure in the centre of a consolidating poroelastic sphere. Later de Leeuw [11] studied an equivalent cylindrical problem, see also Verruijt [22]. The Mandel-Cryer effect has been confirmed by laboratory experiments (Gibson, Knight & Taylor [12] and Verruijt [24]), and field tests (the Noordbergum effect (Verruijt [22], Rodrigues [19])).

The purpose of this paper is to gain a better understanding of the Mandel-Cryer effect. We explain by means of rigorous mathematical techniques the reason of this non-monotone pressure behaviour.

Starting point is the setting in which both fluid and grains are incompressible, the porous medium is homogeneous and isotropic and gravity can be disregarded. Then, as in Coussy [6] or Verruijt [22], the fluid mass balance reads

$$\left.\begin{aligned} &\partial_t \mathcal{E} + \operatorname{div} \mathbf{q} = 0, && (1.2)\\ &\mathcal{E} = \operatorname{div} \mathbf{u}, && (1.3)\\ &\mathbf{q} = -\frac{K}{\eta_f}\nabla p, && (1.4) \end{aligned}\right\} \text{ in } \ \Omega \ \text{ and for } t > 0,$$

where $\mathcal{E}$ $[-]$ denotes volume strain, $\mathbf{q} = (q_x, q_y)$ $[m/s]$ fluid discharge, $\mathbf{u} = (u_x, u_z)$ $[m]$ skeleton displacement, K $[m^2]$ intrinsic scalar permeability, η_f $[Pas]$ fluid viscosity and $p[Pa]$ fluid pressure. Concerning the notation, ∂_* denotes the partial derivative with respect to $*$ and B_A the A-th component of the (vectorial or tensorial) entry B.

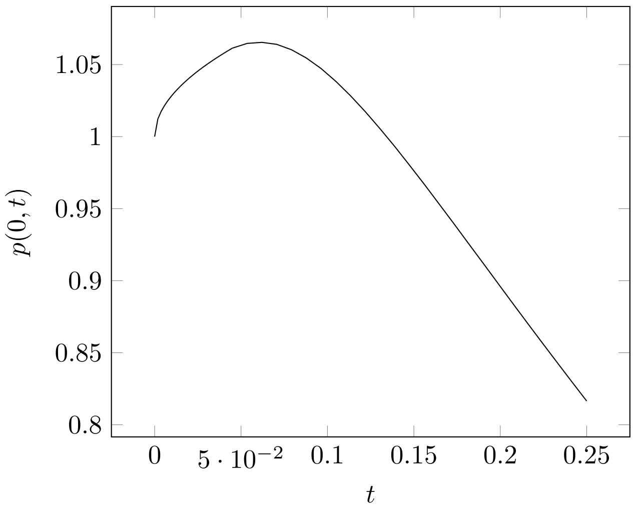

FIG. 2. *Behavior of dimensionless pressure at the centre of the sample as a function of dimensionless time, showing the Mandel-Cryer effect. Here Poisson's ratio is $\nu = 1/3$, $(\lambda + 2\mu)/\mu = 4$. This curve is constructed from a Laplace Transform based approximation for small t and the Fourier approximation (2.27)-(2.30) for larger values of t.*

The momentum balance is given by Biot's formulation (Biot [5]),

$$-\text{div}\ \sigma = 0, \tag{1.5}$$

$$\sigma = 2\mu e(\mathbf{u}) + (\lambda \mathcal{E} - \alpha p)\mathbb{I}. \tag{1.6}$$

Here σ[Pa] is the total stress tensor, μ[Pa] and λ[Pa] the Lamé parameters, $e(\mathbf{u})$ $[-]$ $= \frac{1}{2}(\nabla \mathbf{u} + \nabla^\tau \mathbf{u})$ the linearized strain tensor, $\mathbb{I}$ the identity tensor and $\alpha \in (0, 1]$ Biot's effective stress parameter. In the engineering literature (Abousleiman et al [2] or Verruijt [22]), one often writes $\alpha = 1 - K_B/K_g$, where K_B is the drained bulk modulus and K_g the bulk modulus of the grains. Since they are assumed incompressible, we have $K_g = \infty$ and thus $\alpha = 1$. Therefore, we replace (1.6) by

$$\sigma = 2\mu e(\mathbf{u}) + (\lambda \mathcal{E} - p)\mathbb{I}. \tag{1.7}$$

Along the boundary of Ω we have for all $t > 0$ the Mandel conditions:

$$\{x = 0\} : \ u_x = 0,\ \sigma_{xz} = 0 \text{ and } \partial_x p = 0; \tag{1.8}$$

$$\{x = a\} : \ \sigma_{xz} = 0,\ \sigma_{xx} = 0 \text{ and } p = 0; \tag{1.9}$$

$$\{z = 0\} : \ u_z = 0,\ \sigma_{xz} = 0 \text{ and } \partial_z p = 0; \tag{1.10}$$

$$\{z = h\} : \ u_z = f(t),\ \int_0^a \sigma_{zz}\, dx = -F,\ \sigma_{xz} = 0 \text{ and } \partial_z p = 0. \tag{1.11}$$

Here, $f(t)$ is the unknown displacement at the top of the sample and F is the total load on Ω.

Initially, at $t = 0$, we have

$$\mathcal{E}|_{t=0} = 0 \quad \text{in} \quad \Omega. \tag{1.12}$$

The plan of the paper is as follows. In Section 2, we reduce the two-dimensional Mandel problem (1.2)-(1.12) to one-dimensional non-standard parabolic problems for the volume strain $\mathcal{E}$ and the pressure p. In this reduction, $\mathcal{E} = \mathcal{E}(x,t)$ and $p = p(x,t)$ only. We further show that (1.12) implies

$$p|_{t=0} = \frac{F}{2a} \quad \text{in } \Omega. \tag{1.13}$$

We present the Fourier expansion for $\mathcal{E}$ and p, and discuss the corresponding Hilbert spaces. In Section 3 we consider the functional analytic setting of the $\mathcal{E}$-problems and show that the Fourier expansion represents its unique solution. In Section 4 it is shown that $\partial_x p(a,t) = O(t^{-1/2})$ and $||\partial_x p(\cdot,t)||_{L^2(0,a)} = O(t^{-1/4})$ as $t \searrow 0$. This corresponds to the numerical findings of Phillips [17] and Phillips & Wheeler [18]. The main result of this section is the demonstration of the Mandel-Cryer effect by means of the inverse distributional Laplace Transform.
The conclusions are presented in Section 5.

2. Mandel problem as non-standard parabolic problem.

In this section we present the main steps of the derivation of Mandel's problem. We follow in essence the work of Abousleiman et al [2], Coussy[6] and Verrujt [22]. Since the plates are rigid, impervious and frictionless with respect to the rock sample and since the lateral boundary conditions are constant, we look for a solution of problem (1.2)-(1.12), that describes a configuration in which horizontal planes in the sample move undistorted downwards ($F > 0$), vertical planes move undistorted sideways and in which the fluid flow is parallel to the plates. In terms of the displacements this means that the vertical component u_z does not depend on x and the horizontal component u_x does not depend on z. Hence, we seek a solution that satisfies

$$\left.\begin{array}{l} u_x = u_x(x,t), \\ u_z = u_z(z,t), \\ q_z = 0, \end{array}\right\} \quad \text{in} \quad \Omega \quad \text{and for } t > 0. \tag{2.1}$$

These assumptions imply

$$\left.\begin{array}{l} \sigma_{xz} = 0, \\ p = p(x,t), \\ e_{xx} = \partial_x u_x = e_{xx}(x,t), \\ e_{zz} = \partial_z u_z = e_{zz}(z,t), \end{array}\right\} \quad \text{in} \quad \Omega \quad \text{and for } t > 0. \tag{2.2}$$

Balancing forces in $x-$direction gives

$$0 = \partial_x \sigma_{xx} + \partial_z \sigma_{xz} = \partial_x \sigma_{xx}.$$

Then boundary condition (1.9) implies

$$\sigma_{xx} = 2\mu e_{xx} + \lambda \mathcal{E} - p = 0, \tag{2.3}$$

and consequently

$$\mathcal{E} = \mathcal{E}(x,t) \quad \text{in} \quad \Omega \quad \text{and for } t > 0. \tag{2.4}$$

Writing (2.3) as

$$(2\mu + \lambda)\mathcal{E} - p = 2\mu e_{zz}, \tag{2.5}$$

we deduce that

$$e_{zz} = e_{zz}(t) \quad \text{in} \quad \Omega \quad \text{and for } t > 0. \tag{2.6}$$

Next consider, using again expression (2.3),

$$\sigma_{zz} = 2\mu e_{zz} + \lambda\mathcal{E} - p = (2\mu+\lambda)\mathcal{E} - p - 2\mu e_{xx} = 2(\mu+\lambda)\mathcal{E} - 2p. \tag{2.7}$$

Hence

$$\sigma_{zz} = \sigma_{zz}(x,t) \quad \text{in} \quad \Omega \quad \text{and for } t>0. \tag{2.8}$$

Integrating (2.5) and (2.7) results in

$$2\mu a e_{zz}(t) = (2\mu+\lambda)\int_0^a \mathcal{E}\,dx - \int_0^a p\,dx$$

and

$$-F = 2(\mu+\lambda)\int_0^a \mathcal{E}\,dx - 2\int_0^a p\,dx.$$

Combining these expressions and (2.5) gives the following relation between the fluid pressure and the volume strain:

$$p = (2\mu+\lambda)\mathcal{E} - \frac{\mu}{a}\int_0^a \mathcal{E}\,dx + \frac{F}{2a}, \tag{2.9}$$

in Ω and for $t>0$. Hence we have the pressure initial condition

$$p|_{t=0^+} = \frac{F}{2a}. \tag{2.10}$$

Substituting (2.9) into equations (1.2), (1.4) and the boundary conditions at $x=0$ and $x=a$, yields the following parabolic problem for the volume strain

$$\partial_t\mathcal{E} - \frac{(\lambda+2\mu)K}{\eta_f}\partial_{xx}\mathcal{E} = 0 \quad \text{for } 0<x<a,\ t>0, \tag{2.11}$$

$$\partial_x\mathcal{E}(0,t) = 0 \quad \text{for } t>0, \tag{2.12}$$

$$p(a,t)=0 \Rightarrow (\lambda+2\mu)\mathcal{E}(a,t) = -\frac{F}{2a} + \frac{\mu}{a}\int_0^a \mathcal{E}(s,t)\,ds \quad \text{for } t>0, \tag{2.13}$$

$$\mathcal{E}(x,0) = 0 \quad \text{for } 0<x<a. \tag{2.14}$$

Using again expression (2.9), this problem can be rewritten straightforwardly in terms of the fluid pressure. Then it reads

$$\partial_t p - \frac{(\lambda+2\mu)K}{\eta_f}\partial_{xx}p = -\frac{\mu K}{a\eta_f}\partial_x p(a,t) \quad \text{for } 0<x<a,\ t>0, \tag{2.15}$$

$$\partial_x p(0,t) = p(a,t) = 0 \quad \text{for } t>0, \tag{2.16}$$

$$p(x,0^+) = \frac{F}{2a} \quad \text{for } 0<x<a. \tag{2.17}$$

REMARK 1. *(i) The pressure boundary condition (2.13) yields a non-local boundary condition for $\mathcal{E}$. In the pressure formulation a source term of unknown strength appears in the right hand side of (2.15). In this respect, both formulations yield non-standard problems.*

(ii) Relation (2.9) expresses p in terms of $\mathcal{E}$ and $\int_0^a \mathcal{E}(s,t)\,ds$. Likewise, a relation can be deduced that expresses $\mathcal{E}$ in terms of p and $\int_0^a p(s,t)\,ds$. It reads

$$(2\mu+\lambda)\mathcal{E} = p + \frac{\mu}{a(\mu+\lambda)}\int_0^a p(s,t)\,ds - \frac{2\mu+\lambda}{\mu+\lambda}\frac{F}{2a}. \tag{2.18}$$

For convenience we introduce the scaling

$$x := \frac{x}{a}, \quad t := \frac{(\lambda + 2\mu)K}{a^2 \eta_f}, \quad p := \frac{2a}{F} p,$$

and the variable

$$\mathcal{E} = \frac{F}{2a\mu}\left(w - \frac{\mu}{\lambda + \mu}\right).$$

Then for w we have the following volume strain problem

$$\left.\begin{array}{lll} (2.19) & \partial_t w = \partial_{xx} w \quad \text{for } 0 < x < 1,\ t > 0, \\ (2.20) & \partial_x w(0,t) = 0, \quad w(1,t) = \dfrac{\mu}{\lambda + 2\mu} \displaystyle\int_0^1 w(s,t)\, ds \quad \text{for } t > 0, \\ (2.21) & w(x,0) = \dfrac{\mu}{\lambda + \mu} \quad \text{for } 0 < x < 1. \end{array}\right\} (PVS)$$

For the scaled pressure we find

$$\left.\begin{array}{lll} (2.22) & \partial_t p = \partial_{xx} p - \dfrac{\mu}{2\mu + \lambda} \partial_x p(1,t) \quad \text{for } 0 < x < 1,\ t > 0, \\ (2.23) & \partial_x p(0,t) = 0, \quad p(1,t) = 0 \quad \text{for } t > 0, \\ (2.24) & p(x,0) = 1 \quad \text{for } 0 < x < 1. \end{array}\right\} (PP)$$

In terms of the scaled variables, relation (2.9) becomes

$$p(x,t) = \frac{\lambda + 2\mu}{\mu} w(x,t) - \int_0^1 w(s,t)\, ds. \tag{2.25}$$

The problem for the volume strain (PVS), with the nonlocal boundary condition at $x = 1$, and for the pressure (PP), with the unknown source term $\partial_x p(1,t)$, was not written as such in the engineering literature. Abousleiman et al [2] and Coussy [6] directly write the problem in terms of a Fourier expansion, while Verruijt[22] writes the pressure equation directly in terms of the Laplace transform.

Remark 2. *(Abousleiman et al [2], Verruijt [22]) The elastic parameters in problems (PVS) and (PP) can be expressed in terms of Poisson's ratio ν:*

$$\frac{\mu}{\lambda + 2\mu} = \frac{1}{2}\frac{1 - 2\nu}{1 - \nu} \quad \textit{and} \quad \frac{\mu}{\lambda + \mu} = 1 - 2\nu.$$

As in Abousleiman et al [2] or Coussy [6], the following Fourier expansions are found as solutions of (PVS) and (PP):

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 t}\, e_n(x) \tag{2.26}$$

and

$$p(x,t) = \frac{\lambda + 2\mu}{\mu} \sum_{n=1}^{\infty} A_n \big(e_n(x) - e_n(1)\big) e^{-\alpha_n^2 t}. \tag{2.27}$$

Here $\{\alpha_n\}_{n=1}^{\infty}$ are the positive roots of

$$\tan\alpha_n = \frac{\lambda+2\mu}{\mu}\alpha_n, \tag{2.28}$$

$$e_n(x) := \cos(\alpha_n x), \tag{2.29}$$

and

$$A_n = 2\frac{\cos\alpha_n}{1-\frac{\lambda+2\mu}{\mu}\cos^2\alpha_n}. \tag{2.30}$$

REMARK 3. *Let $\gamma_n = (n-1/2)\pi - \alpha_n$. Then one verifies that*

$$\gamma_n > 0, \quad \gamma_{n+1} < \gamma_n < \dots\dots < \gamma_1 \in (0,\pi/2) \quad \textit{for } n \in \mathbb{N}$$

and $\lim_{n\to\infty}\gamma_n = 0$. Consequently, the denominator in (2.30) is strictly positive.

The numbers $\{\beta_n = \alpha_n^2\}_{n=1}^{\infty}$ and the functions $\{e_n\}_{n=1}^{\infty}$ are eigenvalues and eigenfunctions of the nonlocal spectral boundary value problem

$$-u'' = \beta u \quad \text{for} \quad 0<x<1, \tag{2.31}$$

$$u'(0)=0, \quad u(1) = \frac{\mu}{\lambda+2\mu}\int_0^1 u\ dx. \tag{2.32}$$

Integrating (2.31) yields

$$-u'(1) = \beta\int_0^1 u\ dx.$$

Hence, the nonlocal boundary condition at $x=1$ can be replaced by

$$-u'(1) = \frac{\lambda+2\mu}{\mu}\beta u(1). \tag{2.33}$$

Multiplying the equation for $\{\beta_n, e_n\}$ by e_m and integrating the result in $(0,1)$ gives

$$\int_0^1 e_n' e_m'\ dx = \beta_n\int_0^1 e_n e_m\ dx + e_n'(1)e_m(1).$$

Using (2.32) and (2.33), this expression can be written as

$$\int_0^1 e_n' e_m'\ dx = \beta_n\{\int_0^1 e_n e_m\ dx - \frac{\mu}{\lambda+2\mu}\int_0^1 e_n\ dx\int_0^1 e_m\ dx\}. \tag{2.34}$$

Similarly,

$$\int_0^1 e_n' e_m'\ dx = \beta_m\{\int_0^1 e_n e_m\ dx - \frac{\mu}{\lambda+2\mu}\int_0^1 e_n\ dx\int_0^1 e_m\ dx\}. \tag{2.35}$$

Next we introduce the space

$$W = \{\ L^2(0,1), \text{ equipped with inner product} \quad <u,v>:= (u,v)_{L^2(0,1)} - \frac{\mu}{\lambda+2\mu}\int_0^1 u\ dx\int_0^1 v\ dx\}.$$

Expressions (2.34)-(2.35) imply that $\{e_n\}_{n=1}^{\infty}$ are orthogonal in W.

Further,

$$||u||_W = \sqrt{< u, u >} \quad \text{is equivalent to} \quad ||u||_{L^2(0,1)},$$

since

$$\frac{\lambda+\mu}{\lambda+2\mu}||u||^2_{L^2(0,1)} \le ||u||^2_W \le ||u||^2_{L^2(0,1)} \tag{2.36}$$

for all $u \in L^2(0,1)$.

Finally we observe that

$$\begin{aligned}(e_n - e_n(1), e_m)_{L^2(0,1)} &= (e_n, e_m)_{L^2(0,1)} - e_n(1)\int_0^1 e_m \; dx \\ &= (e_n, e_m)_{L^2(0,1)} - -\frac{\mu}{\lambda+2\mu}\int_0^1 e_n \; dx \int_0^1 e_m \; dx \\ &=< e_n, e_m >, \end{aligned} \tag{2.37}$$

This equality implies that the expansion of the volume strain in W is equivalent to the expansion of the pressure in $L^2(0,1)$, since

$$\begin{aligned}\frac{\mu}{\lambda+2\mu}(e_m, 1)_{L^2(0,1)} = \sum_{n=1}^{+\infty} A_n(e_n - e_n(1), e_m)_{L^2(0,1)} = \sum_{n=1}^{+\infty} A_n < e_n, e_m >= \\ A_m||e_m||^2_W = \frac{\mu}{\lambda+\mu} < e_m, 1 > .\end{aligned} \tag{2.38}$$

3. Functional analytic setting. At this point, it is not clear if $\{e_n\}_{n=1}^{\infty}$ is really a basis for W and if $\{\beta_n\}_{n=1}^{\infty}$ is the entire spectrum. For this reason we give a rigorous mathematical argument that completes the computations.

To recast eigenvalue problem (2.31)- (2.33) in an abstract framework we introduce the space

$$V = \{ \; u \in H^1(0,1) : \; u(1) - \frac{\mu}{\lambda+2\mu}\int_0^1 u \; dx = 0\}. \tag{3.1}$$

Clearly, V is a closed subspace of $H^1(0,1)$.

Based on (2.34)-(2.35), we consider the variational formulation:

Find $u \in V$ and $\beta \in \mathbb{R}$, $u \neq 0$, such that

$$\int_0^1 u'(x)\varphi'(x) \; dx = \beta\{\int_0^1 u(x)\varphi(x) \; dx - \frac{\mu}{\lambda+2\mu}\int_0^1 u(x) \; dx \int_0^1 \varphi(x) \; dx\}, \quad \forall \varphi \in V. \tag{3.2}$$

Then we have

LEMMA 3.1. *Any solution $\{u, \beta\}$ of (3.2) solves problem (2.31)- (2.33).*

Proof. Let $\{u, \beta\}$ satisfy (3.2) and let $\varphi \in C_0^\infty(0,1)$ with $\int_0^1 \varphi \; dx = 0$. Then $\varphi \in V$ and from (3.2),

$$\int_0^1 u'(x)\varphi'(x) \; dx = \beta \int_0^1 u(x)\varphi(x) \; dx,$$

or, in distributional sense,

$$< -u'' - \beta u, \varphi >_{\mathcal{D}'(0,1)} = 0 \quad \forall \varphi \in C_0^\infty(0,1), \quad \int_0^1 \varphi \; dx = 0.$$

This implies, see [8, Appendix "Distributions"],

$$-u'' - \beta u = C (= \text{constant}) \quad \text{in} \quad (0,1). \tag{3.3}$$

Hence $u \in C^\infty[0,1]$. Again from (3.2), after integration by parts,

$$u'(1)\varphi(1) - u'(0)\varphi(0) = \int_0^1 (u'' + \beta u)\varphi \; dx - \beta \frac{\mu}{\lambda + 2\mu} \int_0^1 u(x) \; dx \int_0^1 \varphi(x) \; dx.$$

Taking $\varphi \in V$, with $\varphi(1) = \dfrac{\mu}{\lambda + 2\mu} \displaystyle\int_0^1 \varphi(x) \; dx = 0$, and using (3.3), we find:

$$u'(0) = 0.$$

Hence for any $\varphi \in V$

$$u'(1)\varphi(1) = -C \int_0^1 \varphi \; dx - \beta \frac{\mu}{\lambda + 2\mu} \int_0^1 u(x) \; dx \int_0^1 \varphi(x) \; dx$$

or

$$u'(1) + \beta \int_0^1 u(x) \; dx + \frac{\lambda + 2\mu}{\mu} C = 0. \tag{3.4}$$

On the other hand, integrating (3.3),

$$u'(1) + \beta \int_0^1 u(x) \; dx + C = 0. \tag{3.5}$$

Then (3.4) and (3.5) imply $C = 0$ and equation (2.31) results. Since $u \in V$, the second condition in (2.32) is fulfilled as well. Integrating (2.31) implies (2.33). ∎

Writing (3.2) as

$$a(u, \varphi) = \beta < u, \varphi > \quad \forall \varphi \in V,$$

we note that

(i) the injection of V into W is continuous, dense and compact;

(ii) a is a continuous bilinear form, which is symmetric and coercive in that sense, see (2.36),

$$a(\varphi, \varphi) + ||\varphi||^2_W \geq \frac{\lambda + \mu}{\lambda + 2\mu} ||\varphi||^2_{H^1(0,1)} = \frac{\lambda + \mu}{\lambda + 2\mu} ||\varphi||^2_V$$

for all $\varphi \in V$.

Assertion (i) is a direct consequence of the fact that any bounded sequence in V has a convergent subsequence in $L^2(0,1)$ (by Rellich's theorem) and that $L^2(0,1)$ and W are equivalent (by (2.36)). Hence, the injection is compact. The inequality between norms of V and W guarantees continuity of the injection. Finally, the density of V in $L^2(0,1)$ follows from the discussion in the proof of Lemma 3.1.

Then the variational spectral theory, see [9, Chapter 8], implies that problem (3.2) has a countable number of eigenvalues $\{\beta_n\}_{n=1}^\infty$ such that $-1 \leq \beta_1 \leq \beta_2 \leq \dots$, with $\beta_n \to \infty$ as $n \to +\infty$. The problem has only discrete eigenvalues and the corresponding eigenfunctions form an orthonormal basis for the space W and a basis for V. Obviously, $\beta_1 \geq 0$ and we set $\alpha_n = \sqrt{\beta_n}$. The boundary condition at $x = 1$, rules out $\beta_1 = 0$. Thus indeed $\{\beta_n^2\}_{n=1}^\infty$ is the entire spectrum and $\{e_n\}_{n=1}^\infty$ is an orthogonal basis in W.

Next we write (PVS) as an abstract variational parabolic problem. Let $T > 0$, arbitrarily chosen, and let V' denote the dual of V. Then it reads

Find $w \in L^2(0,T;V) \cap C([0,T];W)$, with $\partial_t w \in L^2(0,T;V')$, such that

$$\frac{d}{dt}(w(t),\varphi)_W + a(w(t),\varphi) = 0, \quad \forall \varphi \in V \quad \text{and for almost all } t \in [0,T]; \tag{3.6}$$

$$w(0) = \frac{\mu}{\lambda+\mu} \in W. \tag{3.7}$$

Theorem 3.2. *The abstract problem (3.6)-(3.7) has a unique solution. It is given by the Fourier expansion (2.26), (2.28)-(2.30).*

Proof. The proof is a direct consequence the properties of the spaces V and W (continuous and dense injection of V in W) and continuity and coercivity of the bilinear form a. Details of the existence and uniqueness proof for the classical abstract variational theory are given in Dautray & Lions [10, Chapter 18] or Wloka [25]. In the existence part of the proof one uses a finite dimensional approximation with respect to the basis $\{e_n\}_{n=1}^{\infty}$ in W. Hence the Fourier expansion applies and w is given by (2.26). This series converges strongly in $L^2(0,T;V) \cap C([0,T];W)$, because the partial sums represent a Cauchy sequence in that space. Since

$$\lim_{t\searrow 0} w(1,t) = \frac{\mu^2}{(\lambda+2\mu)(\lambda+\mu)} \neq \frac{\mu}{\lambda+\mu}$$

a Gibbs effect near the corner point $(x=1, t=0)$ may occur. □

Corollary 3.3. *Rescaled and shifted volume strain w satisfies* $w \in C^\infty\big([\delta,T]\times[0,1]\big)$, $\forall \delta > 0$.

4. The Mandel-Cryer effect.

The purpose of this section is to demonstrate rigorously the Mandel-Cryer effect: i.e. the increase of the pressure in the center of the sample, at $\{x=0\}$, for small times.

Let us first consider the pressure equation (2.22). Its unique solution is given by (2.25), where w is the Fourier series (2.26), or directly by the modified Fourier series (2.27). Using (2.27)-(2.30) we compute

$$\partial_x p(1,t) = -2\sum_{n=1}^{\infty} \frac{\sin^2\alpha_n}{1-\frac{\lambda+2\mu}{\mu}\cos^2\alpha_n} e^{-\alpha_n^2 t} < 0 \tag{4.1}$$

and

$$\frac{d}{dt}\big(\partial_x p(1,t)\big) = 2\sum_{n=1}^{\infty} \frac{\alpha_n^2\sin^2\alpha_n}{1-\frac{\lambda+2\mu}{\mu}\cos^2\alpha_n} e^{-\alpha_n^2 t} > 0, \tag{4.2}$$

for all $t > 0$. Hence $-\partial_x p(1,t)$ acts as a source term in (2.22), whose strength decreases in time. Thus one might expect a pressure increase for small $t > 0$. On the other hand, integrating equation (2.22) in $x \in (0,1)$, gives for any $t > 0$,

$$\frac{d}{dt}\int_0^1 p(x,t)\, dx = \frac{\lambda+\mu}{\lambda+2\mu}\partial_x p(1,t) < 0. \tag{4.3}$$

Hence the mean decreases in time. Furthermore, directly from Fourier series (2.27),

$$\lim_{t\to\infty} p(x,t) = 0, \quad \text{uniformly in} \quad 0 \le x \le 1. \tag{4.4}$$

Therefore the behavior of the pressure is a priori not clear and needs to be investigated.

We consider the Laplace transform of the pressure as starting point. Note that the Laplace transform was also used in the work of Verruijt [22].

We first consider the transformed volume strain, i.e. $\mathcal{L}(w(x,t)) = \overline{w}(x,s)$ with $s > 0$, satisfying equation (2.19), initial condition (2.21) and from (2.20) the Neumann condition at $x = 0$. This yields

$$\begin{cases} \dfrac{d^2}{dx^2}\overline{w} = s\overline{w} - \dfrac{\mu}{\mu+\lambda} \quad \text{for } 0 < x < 1, \\ \dfrac{d}{dx}\overline{w}|_{x=0} = 0. \end{cases}$$

and thus

$$\overline{w}(x,s) = C\cosh(x\sqrt{s}) + \frac{\mu}{\mu+\lambda}\frac{1}{s} \tag{4.5}$$

for $0 < x < 1$ and $s > 0$. Here C is a constant to be determined below. Using (2.25), the Laplace transform of the pressure reads

$$\overline{p}(x,s) = \frac{\lambda+2\mu}{\mu}\overline{w}(x,s) - \int_0^1 \overline{w}(y,s)\ dy.$$

Substituting (4.5) gives

$$\overline{p}(x,s) = \frac{1}{s} + C\frac{\lambda+2\mu}{\mu}\cosh(x\sqrt{s}) - \frac{C}{\sqrt{s}}\sinh\sqrt{s}$$

for $0 < x < 1$ and $s > 0$. Now choosing C such that $\overline{p}(1,s) = 0$, yields

$$\overline{p}(x,s) = \frac{1}{s} + \frac{1}{s}\frac{\frac{\lambda+2\mu}{\mu}\cosh(x\sqrt{s}) - \frac{1}{\sqrt{s}}\sinh\sqrt{s}}{\frac{1}{\sqrt{s}}\sinh\sqrt{s} - \frac{\lambda+2\mu}{\mu}\cosh\sqrt{s}}. \tag{4.6}$$

Thus

$$\mathcal{L}\big(p(x,t) - \chi_{\mathbb{R}^+}\big) = \frac{1}{s}\frac{\frac{\lambda+2\mu}{\mu}\cosh(x\sqrt{s}) - \frac{1}{\sqrt{s}}\sinh\sqrt{s}}{\frac{1}{\sqrt{s}}\sinh\sqrt{s} - \frac{\lambda+2\mu}{\mu}\cosh\sqrt{s}}, \tag{4.7}$$

where

$$\chi_{\mathbb{R}^+}(t) = \begin{cases} 1, & \text{for } t > 0, \\ 0, & \text{for } t < 0. \end{cases}$$

4.1. Behaviour of $\partial_x p(1,t)$ as $t \searrow 0$. Expression (4.7) implies

$$\mathcal{L}\big(\partial_x p(x,t)\big) = \frac{1}{\sqrt{s}}\frac{\frac{\lambda+2\mu}{\mu}\sinh(x\sqrt{s})}{\frac{1}{\sqrt{s}}\sinh\sqrt{s} - \frac{\lambda+2\mu}{\mu}\cosh\sqrt{s}} \tag{4.8}$$

for $0 \le x \le 1$, and thus

$$\mathcal{L}\big(\partial_x p(1,t)\big) = \frac{1}{\sqrt{s}}\frac{\frac{\lambda+2\mu}{\mu}\sinh\sqrt{s}}{\frac{1}{\sqrt{s}}\sinh\sqrt{s} - \frac{\lambda+2\mu}{\mu}\cosh\sqrt{s}} \tag{4.9}$$

for all $s > 0$. After some rearrangements, expression (4.9) can be written as

$$\mathcal{L}\big(\partial_x p(1,t)\big) = -\frac{1}{\sqrt{s}} + G(s)$$

or

$$\mathcal{L}\big(\partial_x p(1,t) + \frac{1}{\sqrt{\pi t}}\chi_{\mathbb{R}^+}\big) = G(s), \tag{4.10}$$

where

$$G(s) = \frac{1}{\sqrt{s}} \frac{\frac{\lambda+2\mu}{\mu}(1-\tanh\sqrt{s}) + \frac{1}{\sqrt{s}}\tanh\sqrt{s}}{-\frac{1}{\sqrt{s}}\tanh\sqrt{s} + \frac{\lambda+2\mu}{\mu}}. \tag{4.11}$$

Since $\dfrac{\lambda+2\mu}{\mu} > 2$, $\dfrac{1}{\sqrt{s}}\tanh\sqrt{s} < 1$, and $\tanh\sqrt{s} < 1$, we have

$$G : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{is well defined.}$$

The following estimates hold.

LEMMA 4.1. *There exists a constant $M > 0$ such that for all $s > 0$*
(i) $sG(s) \le M$;
(ii) $|\dfrac{d}{ds}\big(\sqrt{s}G(s)\big)| \le Ms^{-3/2}$.

Proof. **(i)** Directly from (4.11)

$$sG(s) < \frac{\dfrac{\lambda+2\mu}{\mu}(1-\tanh\sqrt{s})\sqrt{s} + \tanh\sqrt{s}}{\dfrac{\lambda+\mu}{\mu}} < \frac{\lambda+2\mu}{\lambda+\mu}\frac{2\sqrt{s}}{e^{2\sqrt{s}}+1} + \frac{\mu}{\lambda+\mu} \le M,$$

where we used $\tanh y < y$ and $\dfrac{y}{e^y+1} < \dfrac{1}{e}$ for all $y > 0$.

(ii) Setting $y = \sqrt{s}$, we note that $(1-\tanh y)y$ and its derivatives have exponential decay as $y \to \infty$. Hence for

$$yG(y^2) = \frac{\frac{\lambda+2\mu}{\mu}(1-\tanh y)y + \tanh y}{-\tanh y + \frac{\lambda+2\mu}{\mu}y}$$

we have

$$|\frac{d}{dy}(yG(y^2))| \le \frac{C}{(\frac{\lambda+2\mu}{\mu} - \frac{\tanh y}{y})^2 y^2} < \big(\frac{\mu}{\lambda+\mu}\big)^2\frac{C}{y^2}.$$

Using now

$$\frac{d}{ds}\big(\sqrt{s}G(s)\big) = \frac{1}{2\sqrt{s}}\frac{d}{d\sqrt{s}}\big(\sqrt{s}G(s)\big)$$

estimate **(ii)** is immediate. ∎

Then we have

LEMMA 4.2. *There exists $M > 0$ such that for all $s > 0$*

$$|s^2\frac{d}{ds}G(s)| \le M.$$

Proof. Using

$$s^{3/2}\frac{d}{ds}\big(\sqrt{s}G(s)\big) = s^2\frac{d}{ds}G(s) + \frac{s}{2}G(s),$$

the estimate is a direct consequence of Lemma 4.1. ∎

We are now in a position to apply the following result, which is due to Prüss (see Arendt et al [3, Corollary 2.5.2])

Prüss's Proposition. *Let X be a Banach space and let $q : \{\Re z > 0\} \to X$ be holomorphic. Then the following holds: if there exists $M > 0$ such that $||zq(z)||_X \le M$ and $||z^2\frac{d}{dz}q(z)||_X \le M$ for $\{\Re z > 0\}$, then there exists a bounded function $f \in C((0,+\infty);X)$ such that*

$$q(z) = \int_0^\infty e^{-zt}f(t)\ dt \quad \text{for} \quad \Re z > 0.$$

In our case $G : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth. In addition, with $s \in \mathbb{C}$,

$$|-\sinh s + \frac{\lambda+2\mu}{\mu}s\cosh s|^2 \ge C_\omega e^{2\Re s}\big((\frac{\lambda+2\mu}{\mu}\Re s - 1)^2 + (\frac{\lambda+2\mu}{\mu}\Im s)^2\big), \quad \forall s,\ \Re s > \omega > 0$$

for some real number ω sufficiently large. Hence G is holomorphic in $\{\Re s > \omega > 0\}$, taking values in $\mathbb{C}$. Thus $X = \mathbb{C}$. Furthermore, the proof of boundedness of the norms $||sG(s)||_X$ and $||s^2\frac{d}{ds}G(s)||_X$ is analogous to real case, but we need that $\Re s > \omega > 0$. Then Prüss's Proposition applies, but with $e^{-\omega t}f(t)$ being bounded.

As a result we have

Proposition 4.3. *There exists $g \in C((0,+\infty);\mathbb{R})$, with $\sup_{t>0}|e^{-\omega t}g(t)| < +\infty$, such that*

$$\partial_x p(1,t) = -\frac{1}{\sqrt{\pi t}} + g(t) \quad \text{for all}\, t > 0.$$

4.2. Behaviour of $t^{1/4}||\partial_x p(\cdot,t)||_{L^2(0,1)}$ as $t \searrow 0$. The singular nature of $\partial_x p(1,t)$ as $t \searrow 0$ clearly influences the behavior of the norm $||\partial_x p(\cdot,t)||_{L^2(0,1)}$ as $t \searrow 0$. This was investigated numerically by Phillips [17] and Phillips & Wheeler [18], who used a combination of mixed and continuous Galerkin finite elements to discretize the poroelasticity equations. They established the behaviour ([17] page 141, formulas (6.1)-(6.2))

$$||\partial_x p(\cdot,t)||^2_{L^2(0,1)} \approx O(t^{-0.488}) \quad \text{and} \quad ||\partial_{xx} p(\cdot,t)||^2_{L^2(0,1)} \approx O(t^{-1.447}), \quad \text{as}\, t \searrow 0.$$

As the following analytical result shows, this is amazingly accurate:

Proposition 4.4. $\lim_{t\searrow 0}||t^{1/4}\partial_x p(\cdot,t)||_{L^2(0,1)} = (2\pi)^{-1/4}$.

Proof. Using Proposition 4.3, we write the pressure equation (2.22) in problem (PP) as

$$\partial_t p - \partial_{xx} p = \frac{\mu}{2\mu+\lambda}\frac{1}{\sqrt{\pi t}} - \frac{\mu}{2\mu+\lambda}g(t) \quad \text{for}\ 0 < x < 1,\ t > 0, \tag{4.12}$$

for some $g \in C([0,1]\times[0,T])$. Thus next to the incompatibility of the initial and boundary conditions at $x = 1$, the right hand side of (4.12) has a $t^{-1/2}$ singularity at $t = 0^+$. Further, we note that the global regularity of p is less than

$$w = \frac{\mu}{\mu+\lambda} + \frac{2a\mu}{F}\mathcal{E} \in L^2(0,T;V) \cap C([0,T],W).$$

The idea is to search for p in the form

$$p(x,t) = u(x,t) + \frac{\mu}{2\mu+\lambda}\sqrt{\frac{t}{\pi}} - v(x,t) - z(x,t), \tag{4.13}$$

where

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_{\eta/2}^{0} e^{-y^2}\, dy, \quad \eta = \frac{x-1}{\sqrt{t}} < 0, \tag{4.14}$$

satisfies

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 \quad \text{in} \quad x<1,\ t>0, \\ u(1,t) = 0 \quad \text{for} \quad t>0, \\ u(x,0) = 0 \quad \text{for} \quad x<1 \end{cases} \tag{4.15}$$

and

$$\frac{2\mu+\lambda}{2\mu}\sqrt{\frac{\pi}{t}}v(x,t) = h(\eta) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\eta/2} e^{-y^2}\big(1-\frac{\eta^2}{4y^2}\big)^{1/2}\, dy. \tag{4.16}$$

The function h is chosen to satisfy the boundary value problem

$$\begin{cases} h'' + \frac{\eta}{2}h' = \frac{1}{2}h \quad \text{for} \quad \eta<0, \\ h(0) = 1, \quad h(-\infty) = 0 \end{cases} \tag{4.17}$$

and its form corresponds to the representation $h(\eta) = U(\frac{3}{2}, \frac{\eta}{\sqrt{2}})e^{-\frac{1}{8}\eta^2}$, where $U(a,z)$ is the parabolic cylinder function (see Abramowitz et al [1, Chapter19] or Temme [20, pages 175-179]). We do not dwell on the subject but only remark than h can be rewritten in the more convenient form

$$h(\eta) = -\frac{\eta}{2\sqrt{\pi}}\int_1^{\infty} e^{-\eta^2 y/4}(y-1)^{1/2}\,\frac{dy}{y}, \quad \eta<0.$$

Then h' is bounded and positive and h'' is positive. A direct computation shows that h satisfies problem (4.17). Finally, $v(x,t) = \frac{2\mu}{2\mu+\lambda}\sqrt{\frac{t}{\pi}}h(\frac{x-1}{\sqrt{t}})$ satisfies

$$\begin{cases} \partial_t v - \partial_{xx} v = 0 \quad \text{in} \quad x<1,\ t>0, \\ v(1,t) = \frac{2\mu}{2\mu+\lambda}\sqrt{\frac{t}{\pi}} \quad \text{for} \quad t>0, \\ v(x,0) = 0 \quad \text{for} \quad x<1. \end{cases} \tag{4.18}$$

Then z is given by

$$\begin{cases} \partial_t z - \partial_{xx} z = \frac{\mu}{2\mu+\lambda}g(t) \quad \text{for} \quad 0<x<1,\ t>0, \\ z(1,t) = 0 \quad \text{and} \quad \partial_x z(0,t) = a(t) \quad \text{for} \quad t>0, \\ z(x,0) = 0 \quad \text{for} \quad 0<x<1. \end{cases} \tag{4.19}$$

Here $a(t) = \partial_x u(0,t) - \partial_x v(0,t) = -\frac{1}{\sqrt{\pi t}}e^{-1/(4t)} - \frac{2\mu}{2\mu+\lambda}\frac{1}{\sqrt{\pi}}h'(-\frac{1}{\sqrt{t}}) \in C^\infty[0,+\infty)$ and $a(0)=0$.

The change of the unknown $z = Z + (x-1)a(t)$ gives

$$(4.20)\quad \begin{cases} \partial_t Z - \partial_{xx} Z = \dfrac{\mu}{2\mu+\lambda} g(t) - (x-1)a'(t) \in C([0,1]\times[0,T]) \quad \text{for} \quad 0<x<1,\ t>0, \\ Z(1,t)=0, \quad \text{and} \quad \partial_x Z(0,t)=0 \quad \text{for} \quad t>0, \\ Z(x,0)=0 \quad \text{for} \quad 0<x<1. \end{cases}$$

We extend Z to an even function $\tilde{Z}$ on $(-1,1)$. Then $\tilde{Z}$ satisfies the heat equation with a continuous in x and t source term on $(-1,1)\times(0,T)$. Next, $\tilde{Z}$ is zero at $x=-1$, $x=1$ and $t=0$. Hence, we are in situation to apply the parabolic regularity theory from Ladyzenskaja et al [14, Chapter4, Theorem 9.1]. It gives $\tilde{Z} \in W_q^{2,1}((-1,1)\times(0,T))$, $\forall q \in (1,+\infty)$. Therefore, $\tilde{Z}$ and $\partial_x \tilde{Z}$ are Hölder continuous in x and t on $[-1,1]\times[0,T]$. The same property holds for z and $\partial_x z$ on $[-1,0]\times[0,T]$.

Finally

$$\lim_{t\searrow 0} ||t^{1/4}\partial_x p(\cdot,t)||_{L^2(0,1)} = \lim_{t\searrow 0} ||t^{1/4}\partial_x u(\cdot,t)||_{L^2(0,1)} = \lim_{t\searrow 0} \frac{t^{-1/4}}{\sqrt{\pi}} ||e^{-(1-x)^2/(4t)}||_{L^2(0,1)} = (2\pi)^{-1/4}$$

and the proposition is proved. ∎

4.3. Non-monotone pressure at $x=0$, $t>0$. Using the Laplace Transform technique from Subsection 4.1, we are now in position to demonstrate the Mandel-Cryer effect. From (4.6) we deduce

$$\mathcal{L}\big(\partial_t p(x,t)\big) = s\overline{p}(x,s) - 1 = \frac{\frac{\lambda+2\mu}{\mu}\cosh(x\sqrt{s}) - \frac{1}{\sqrt{s}}\sinh\sqrt{s}}{\frac{1}{\sqrt{s}}\sinh\sqrt{s} - \frac{\lambda+2\mu}{\mu}\cosh\sqrt{s}},$$

for $s>0$. Again after a bit of rearrangement we obtain at $x=0$:

$$\mathcal{L}\big(\partial_t p(0,t)\big) = \frac{1}{\frac{\lambda+2\mu}{\mu}\sqrt{s}-1} + Q(s),$$

for $s > \omega > (\dfrac{\mu}{\lambda+2\mu})^2$, where

$$Q(s) = \frac{\frac{\lambda+2\mu}{\mu}\sqrt{s}(\cosh\sqrt{s} - \sinh\sqrt{s}) + (\frac{\lambda+2\mu}{\mu})^2 s - \frac{\lambda+2\mu}{\mu}\sqrt{s}}{(\sinh\sqrt{s} - \frac{\lambda+2\mu}{\mu}\sqrt{s}\cosh\sqrt{s})(\frac{\lambda+2\mu}{\mu}\sqrt{s}-1)}.$$

Since $Q(s)$ is exponentially small in $\sqrt{s}$, we do not have to keep it in the estimates. As in Subsection 4.1,

$$(4.21)\quad \mathcal{L}\big(\partial_t p(0,t) - \frac{\mu}{\lambda+2\mu}\frac{1}{\sqrt{\pi t}}\chi_{\mathbb{R}^+}\big) = \frac{\frac{\mu}{\lambda+2\mu}}{\sqrt{s}(\frac{\lambda+2\mu}{\mu}\sqrt{s}-1)} + Q(s),$$

for $s > \omega > (\dfrac{\mu}{\lambda+2\mu})^2$.

One easily verifies that

(i) $\displaystyle \lim_{s\to\infty} s\frac{\frac{\mu}{\lambda+2\mu}}{\sqrt{s}(\frac{\lambda+2\mu}{\mu}\sqrt{s}-1)} = (\frac{\mu}{\lambda+2\mu})^2$;

(ii) $\displaystyle |s\frac{\frac{\mu}{\lambda+2\mu}}{\sqrt{s}(\frac{\lambda+2\mu}{\mu}\sqrt{s}-1)}| \le M \quad \text{for} \quad s>\omega>(\frac{\mu}{\lambda+2\mu})^2.$;

(iii) $\displaystyle |s^2\frac{d}{ds}\frac{\frac{\mu}{\lambda+2\mu}}{\sqrt{s}(\frac{\lambda+2\mu}{\mu}\sqrt{s}-1)}| \le M \quad \text{for} \quad s>\omega>(\frac{\mu}{\lambda+2\mu})^2.$;

for some $M > 0$.

Again applying Prüss's Proposition, we have

PROPOSITION 4.5. *There exists $q \in C((0,+\infty),\mathbb{R})$, with $\sup_{t>0} |e^{-\omega t} q(t)| < \infty$, $\omega > 0$, such that*

$$\partial_t p(0,t) = \frac{\mu}{\lambda + 2\mu} \frac{1}{\sqrt{\pi t}} + q(t) \quad \textit{for all}\, t > 0.$$

Since q is bounded on $[0,T]$, for every $T > 0$, this result implies that there exists $t_0 > 0$ such that

$$\partial_t p(0,t) > 0 \quad \text{for} \quad 0 < t < t_0,$$

yielding the Mandel-Cryer effect.

5. Conclusion. In this paper we consider Mandel's problem (Mandel [15]) in poroelasticity. This problem describes the behaviour of a water saturated porous slab, that is subjected to a symmetrical load at top and bottom while water is drained from the lateral sides, see Figure 1. It was observed that the fluid pressure in the center of the sample first increases in time and decreases later. The behaviour is known as the Mandel-Cryer effect. Mandel's problem received significant attention in the engineering literature, because it admits an explicit solution for the volume strain $\mathcal{E}$ and the pore pressure p. These solutions are given in terms of Fourier series.

We give a rigorous mathematical foundation of the Mandel problem. To this end we first formulate non-standard parabolic problems for $\mathcal{E}$ and p. The $\mathcal{E}-$problem has a nonlocal boundary condition at the outflow boundary $x = 1$ (equations (2.19)-(2.21)), the $p-$problem has a source term of unknown strength (equations (2.22)-(2.24)). The volume strain and the pore pressure are related through (2.25). We construct a Fourier approximation for $\mathcal{E}$ (and for p) and show that the corresponding Hilbert space, taking into account the nonlocal boundary condition at $x = 1$, is

$$W = \big\{L^2(0,1), \text{ with inner product } \quad < u, v >= (u,v)_{L^2(0,1)} - \frac{\mu}{\lambda + 2\mu} \int_0^1 u\, dx \int_0^1 v\, dx\big\},$$

i.e. the eigenfunctions of the corresponding spectral problem form an orthogonal basis in W. We show, that the Fourier series converges strongly in $L^2(0,T;V) \cap C([0,T];W)$,where the space V is defined in (3.1).

The main result is the mathematical proof of the Mandel-Cryer effect. Here we use Laplace Transform techniques applied to the pressure equation. In particular, we formulate the transformed pressure in such way, that a fundamental result of Prüss [3] can be used.

We first investigate the singular behaviour of the source term in (2.22), for which we obtain

$$\partial_x p(1,t) + \frac{1}{\sqrt{\pi t}} = O(1), \quad \text{as}\, t \to 0+\,. \tag{5.1}$$

We show that this implies

$$||t^{1/4} \partial_x p(\cdot,t)||_{L^2(0,1)} = (2\pi)^{-1/4} \quad \text{as} \quad t \to 0+\,. \tag{5.2}$$

The exponent 1/4 is confirmed by the numerical results of Phillips [17] and Phillips & Wheeler [18], who found numerically the exponent 0.244.

Using again the result of Prüss [3], it follows that there exists $q \in C((0,\infty);\mathbb{R})$, with $e^{-\omega t} q$ bounded in $\mathbb{R}^+$ for some $\omega > 0$, such that

$$\partial_t p(0,t) = \frac{\mu}{\lambda + 2\mu} \frac{1}{\sqrt{\pi t}} + q(t) \quad \text{for all}\, t > 0. \tag{5.3}$$

From this expression the increase of the pressure for small times is immediate.

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