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An Introduction to Conservation Laws: Theory and Applications to Multi-Phase Flow

Preface

These lecture notes result from courses given at Leiden University and at Delft University of Technology. They are intended for students in applied mathematics, physics, fluid mechanics and the engineering sciences.

The material used is taken from existing literature as it developed over the past four decades. It could not have been written without J. Smoller's "Shock Waves and Reaction-Diffusion Equations" and R. LeVeque's "Numerical Methods for Conservation Laws". The result is fairly self-contained and is meant as an introduction to the exciting area of conservation laws.

Conservation laws in the form of hyperbolic first order partial differential equations arise in a wide variety of models describing transport phenomena. Generally, they result if dissipative – second-order parabolic – terms are disregarded. For instance, the well-known Buckley–Leverett equation in two-phase porous media flow arises when capillary forces are absent, see Appendix A. At first sight, a much simpler set of equations (or equation) results which should describe the behaviour of the system in that limit (of vanishing viscous or capillary forces). Indeed, discontinuous solutions in the form of shock waves occur, as to be expected from the simplifying assumptions. Here, however, one of the main difficulties and issues in the theory of conservation laws enters. By considering 'conservation' only, multiple solutions are possible for a given initial-boundary value problem and additional conditions are needed to select the unique and physically correct solution.

Such uniqueness conditions are called *entropy conditions*. The word 'entropy' is used because much of the theory of conservation laws originates from the equations of gas dynamics where entropy plays its natural role. Entropy conditions can be derived from the "full" problem, i.e. including the small dissipative terms, by passing to the limit. For important classes of problems they are independent of the form of the dissipative terms and they can be stated directly in terms of quantities appearing in – or related to – the conservation law itself. Therefore one 'forgets' the limit process of vanishing viscosity (Burgers equation) and considers the equation directly. For scalar equations with dissipative terms in divergence form this can easily be made rigorous. For systems, however, this is in general not true and research is still in progress to understand and quantify the dependence of the entropy conditions on the form and nature of the vanishing dissipative terms.

Much of the above is treated in these lecture notes. Part I is concerned with the scalar case and treats

Burgers equation, Oleinik's uniqueness and existence proof for convex flux functions, travelling waves and Kruzkov's formulation for general flux functions. Part II is mainly devoted to the Riemann problem for systems. It explains the construction of solutions as well as various theoretical aspects, such as existence of weak shocks, uniqueness, entropy formulation and conditions and viscous profiles of shocks.

I am greatly indebted to Hans Bruining from the Department of Reservoir Engineering of Delft University of Technology. Hans introduced me to the area of conservation laws, patiently explained his multi-phase flow problems in porous media, and convinced me to give a course for the applied mathematicians and petroleum engineers. He also supplied much of the material that led to the chapter on multi-phase flow in porous media.

I am also indebted to the students in Leiden and Delft for their interest and motivating discussions. I am particularly appreciative of the help of Arjan Straathof and Gert-Jan Pieters who converted my handwritten notes into this LaTeX edition.

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Part I

Scalar case

1 The viscous Burgers equation

1.1 Travelling waves

Let $\nu > 0$ and u_{r}, $u_{\mathrm{l}} \in \mathbb{R}$. We consider travelling wave solutions of the equation

$$u_t + \left(\frac{u^2}{2}\right)_x = \nu u_{xx} \qquad \text{for } (x,t) \in \mathrm{Q} := \mathbb{R} \times \mathbb{R}^+ \tag{1.1}$$

which satisfy

$$u(-\infty, t) = u_{\mathrm{l}} \;, \quad u(+\infty, t) = u_{\mathrm{r}} \qquad \text{for all } t > 0 \;. \tag{1.2}$$

To find a travelling wave we set

$$u(x,t) = f(\eta) \qquad \text{with} \quad \eta = x - ct \;.$$

Here $c \in \mathbb{R}$ denotes the wavespeed. Equation (1.1) and the boundary conditions (1.2) imply that f and c should satisfy the boundary value problem

$$-\,cf' + \left(\frac{f^2}{2}\right)' = \nu f'' \qquad \text{in } \mathbb{R} \;, \tag{1.3}$$

$$f(-\infty) = u_{\mathrm{l}} \;, \qquad f(+\infty) = u_{\mathrm{r}} \;. \tag{1.4}$$

Equation (1.3) can be integrated to yield

$$-cf + \frac{f^2}{2} = \nu f' + A \qquad \text{in } \mathbb{R} \;, \tag{1.5}$$

where the constants c and A are determined from the boundary conditions (1.4):

$$\eta \to \infty \qquad \Rightarrow \quad -cu_{\mathrm{r}} + \frac{u_{\mathrm{r}}^2}{2} = A$$

and

$$\eta \to -\infty \quad \Rightarrow \quad -cu_{\mathrm{l}} + \frac{u_{\mathrm{l}}^2}{2} = A \;.$$

This gives

$$c = \tfrac{1}{2}(u_\mathrm{l} + u_\mathrm{r}) \qquad \text{and} \qquad A = -\frac{u_\mathrm{l} u_\mathrm{r}}{2}$$

and for f the equation

$$2\nu f' = (f - u_\mathrm{l})(f - u_\mathrm{r}) \qquad \text{in } \mathbb{R}\ .$$

From this we deduce

- $u_\mathrm{l} < u_\mathrm{r}, \quad f \in (u_\mathrm{l}, u_\mathrm{r}) \quad \Rightarrow \quad f' < 0$ and hence no travelling wave exists.
- $u_\mathrm{l} > u_\mathrm{r}, \quad f \in (u_\mathrm{r}, u_\mathrm{l}) \quad \Rightarrow \quad f' < 0$ and the solution is found by direct integration:

$$f(\eta) = u_\mathrm{r} + (u_\mathrm{l} - u_\mathrm{r}) \bigg/ \left(1 + \exp\left\{\frac{u_\mathrm{l} - u_\mathrm{r}}{2\nu}\eta\right\}\right)\ .$$

This implies

$$u(x,t) = u_\mathrm{r} + (u_\mathrm{l} - u_\mathrm{r}) \bigg/ \left(1 + \exp\left\{\frac{u_\mathrm{l} - u_\mathrm{r}}{2\nu}(x - ct)\right\}\right)$$

for $-\infty < x < \infty$ and $t > 0$. Note that $u \in C^\infty(Q)$ and that

$$u^0(x,t) := \lim_{\nu \downarrow 0} u(x,t) = \begin{cases} u_\mathrm{l} & x < ct \\ (u_\mathrm{l} + u_\mathrm{r})/2 & x = ct \\ u_\mathrm{r} & x > ct\ . \end{cases}$$

Thus the limit u^0 has a discontinuity (a shock), which travels along the curve $x(t) = ct = \dfrac{u_\mathrm{r} + u_\mathrm{l}}{2}t$, see Figure 1.1. Observe that this construction is only possible when $u_\mathrm{l} > u_\mathrm{r}$.

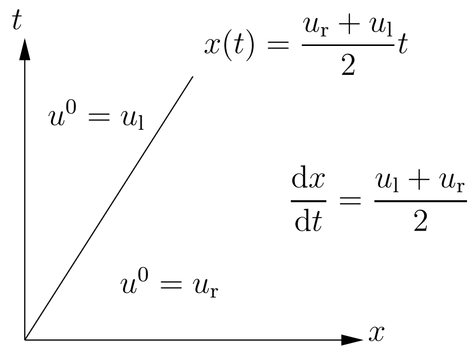


Figure 1.1. *The shock curve*

1.2 A single hump

Consider the problem

$$\begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = \nu u_{xx} & \text{in } Q \\ \\ u(\cdot, 0) = M\delta(\cdot) + u_0 & \text{in } \mathbb{R}\ , \end{cases}$$

where δ denotes the Dirac-distribution at $x = 0$ and where $M > 0$ and $u_0 \in \mathbb{R}$. Without loss of generality we may set $u_0 = 0$. This follows from the transformation

$$\overline{x} = x - u_0 t \qquad \text{and} \qquad \overline{u} = \overline{u}(\overline{x}, t) = u(\overline{x} + u_0 t, t) - u_0 \ ,$$

which implies for $\overline{u}$

$$\begin{cases} \overline{u}_t + \left(\dfrac{\overline{u}^2}{2}\right)_{\overline{x}} = \nu \overline{u}_{\overline{x}\overline{x}} & \text{in } Q \\ \\ \overline{u}(\cdot, 0) = M\delta(\cdot) & \text{in } \mathbb{R} \ . \end{cases}$$

The solution of this problem is called the fundamental solution (of the viscous Burgers equation). It satisfies the equation for all $t > 0$ and the conditions

$$(i) \quad \int_{\mathbb{R}} u(x,t)dx = M \qquad \text{for all } t > 0 \ ,$$

$$(ii) \quad \lim_{t \downarrow 0} u(x,t) = 0 \qquad \text{for all } x \neq 0 \ .$$

To find the fundamental solution we try the self-similar form

$$u(x,t) = t^{\alpha} \varphi(\eta) \quad \text{where} \quad \eta = x t^{\beta} \ ,$$

and where $\alpha, \beta \in \mathbb{R}$ have to be determined from the differential equation. Substitution gives

$$\alpha\varphi + \beta\eta\varphi' + t^{\alpha+\beta+1} \left(\frac{\varphi^2}{2}\right)' = \nu t^{1+2\beta} \varphi'' \ .$$

Choosing

$$\alpha = \beta = -\frac{1}{2} \ ,$$

eliminates the powers of t from the equation and gives

$$-\varphi - \eta\varphi' + \left(\varphi^2\right)' = 2\nu\varphi'' \quad \text{in } \mathbb{R} \ . \tag{1.6}$$

If a non-trivial integrable solution of (1.6) exists, then it induces a solution u of equation (1.1) which satisfies for all $t > 0$

$$\int_{\mathbb{R}} u(x,t) \mathrm{d}x = \int_{\mathbb{R}} t^{-\frac{1}{2}} \varphi\left(x t^{-\frac{1}{2}}\right) \mathrm{d}x = \int_{\mathbb{R}} \varphi(\eta) \mathrm{d}\eta \ .$$

This implies *conservation of mass*. Because of (i) we therefore impose

$$\int_{\mathbb{R}} \varphi(\eta) \mathrm{d}\eta = M \ . \tag{1.7}$$

To satisfy (ii) we impose in addition

$$\lim_{|\eta| \to \infty} \eta\varphi(\eta) = 0 \ . \tag{1.8}$$

Equation (1.6) can be integrated. This gives

$$-\eta\varphi + \varphi^2 = 2\nu\varphi' + A \ .$$

Condition (1.8) implies $A=0$ (check !). Hence we are left with the first order equation

$$\varphi' + \frac{\eta}{2\nu}\varphi = \frac{\varphi^2}{2\nu} \quad \text{in } \mathbb{R}\,.$$

Note that $\varphi' = 0$ as $\varphi = \eta$, see also Figure 1.2 Using an appropriate integrating factor, this equation can be solved explicitly. Since

$$\left(\exp\left\{\frac{\eta^2}{4\nu}\right\}\varphi\right)' = \frac{1}{2\nu}\exp\left\{\frac{\eta^2}{4\nu}\right\}\varphi^2\,,$$

we find for

$$\psi = \exp\left\{\frac{\eta^2}{4\nu}\right\}\varphi$$

the equation

$$\psi' = \frac{1}{2\nu}\exp\left\{-\frac{\eta^2}{4\nu}\right\}\psi^2\,, \qquad \text{or} \qquad -\left(\frac{1}{\psi}\right)' = \frac{1}{2\nu}\exp\left\{-\frac{\eta^2}{4\nu}\right\}\,.$$

Hence ψ is given by

$$\psi(\eta) = \frac{1}{K - \dfrac{1}{2\nu}\displaystyle\int_0^\eta \exp\left\{-\frac{s^2}{4\nu}\right\}\mathrm{d}s}\,,$$

and consequently we find for φ

$$\varphi(\eta) = \exp\left\{-\frac{\eta^2}{4\nu}\right\}\Big/\left\{K - \frac{1}{2\nu}\int_0^\eta \exp\left\{-\frac{s^2}{4\nu}\right\}\mathrm{d}s\right\}\,,$$

or

$$\varphi(\eta) = \exp\left\{-\frac{\eta^2}{4\nu}\right\}\Big/\left\{K + \frac{1}{\sqrt{\nu}}\int_{\frac{\eta}{2\sqrt{\nu}}}^\infty \exp\left\{-s^2\right\}\mathrm{d}s\right\}$$

after a suitable redefinition of K. This constant of integration is determined by condition (1.7). The result is (verify !)

$$K = \frac{\sqrt{\dfrac{\pi}{\nu}}}{\exp\left\{\dfrac{M}{2\nu}\right\} - 1}$$

and thus

$$\varphi(\eta) = \frac{\sqrt{\nu}\left(\exp\left\{\dfrac{M}{2\nu}\right\} - 1\right)\exp\left\{\dfrac{-\eta^2}{4\nu}\right\}}{\sqrt{\pi} + \left(\exp\left\{\dfrac{M}{2\nu}\right\} - 1\right)\displaystyle\int_{\frac{\eta}{2\sqrt{\nu}}}^\infty \exp\left\{-s^2\right\}\mathrm{d}s}\,, \tag{1.9}$$

see Figure 1.2.

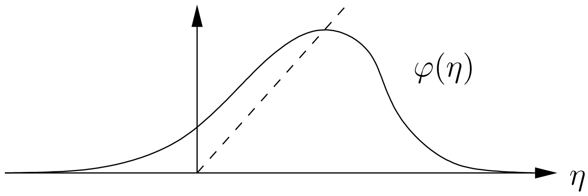


Figure 1.2. *The graph of* φ

Remark 1.1. *The similarity solution* $u(x,t)=t^{-\frac{1}{2}}\varphi\left(xt^{-\frac{1}{2}}\right)$ *satisfies for all* $t>0$

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} = \begin{cases} t^{\frac{1-p}{2p}}\|\varphi\|_{L^p(\mathbb{R})} & \textit{for } 1 \leqslant p < \infty \\ t^{-\frac{1}{2}}\|\varphi\|_{L^\infty(\mathbb{R})} & \textit{for } p = \infty\ . \end{cases}$$

Next we study the behaviour of the solution when $\nu\downarrow 0$. First consider $\eta\leqslant 0$:

$$0 < \varphi(\eta) < \sqrt{\nu}\frac{\exp\left\{\dfrac{M}{2\nu} - \dfrac{\eta^2}{4\nu}\right\}}{\exp\left\{\frac{M}{2\nu}\right\}\displaystyle\int_{\frac{\eta}{2\sqrt{\nu}}}^{\infty}\exp\left\{-s^2\right\}\mathrm{d}s} < 2\sqrt{\frac{\nu}{\pi}}\exp\left\{-\frac{\eta^2}{4\nu}\right\}\ .$$

Thus $\varphi(\eta)\to 0$ as $\nu\downarrow 0$, uniformly in $\eta\in(-\infty,0]$. Next consider $\eta\geqslant\sqrt{2M}$:

$$0 < \varphi(\eta) < \sqrt{\frac{\nu}{\pi}}\exp\left\{\frac{M}{2\nu} - \frac{\eta^2}{4\nu}\right\}\ .$$

Thus $\varphi(\eta)\to 0$ as $\nu\downarrow 0$, uniformly in $\eta\in[\sqrt{2M},\infty)$. Finally let $0<\eta<\sqrt{2M}$. We now use the estimate (ABRAMOWITZ & STEGUN [1])

$$\exp\left\{z^2\right\}\int_z^\infty \exp\left\{-s^2\right\}\mathrm{d}s \to \frac{1}{2z} \qquad \text{as } z\to\infty\ .$$

This gives

$$\varphi(\eta) \sim \sqrt{\nu}\frac{\exp\left\{\dfrac{M}{2\nu} - \dfrac{\eta^2}{4\nu}\right\}}{\sqrt{\pi} + \exp\left\{\dfrac{M}{2\nu}\right\}\dfrac{\sqrt{\nu}}{\eta}\exp\left\{-\dfrac{\eta^2}{4\nu}\right\}} \qquad \text{for small } \nu\ .$$

Thus $\varphi(\eta)\to\eta$ as $\nu\downarrow 0$, pointwise in $\eta\in(0,\sqrt{2M})$. In terms of the original variables x and t we have as limit profile

$$u^0(x,t) := \lim_{\nu\downarrow 0} u(x,t) = \begin{cases} 0 & x\leqslant 0\ , \\ x/t & 0<x<\sqrt{2Mt}\ , \\ 0 & x\geqslant\sqrt{2Mt}\ , \end{cases}$$

see Figure 1.3.

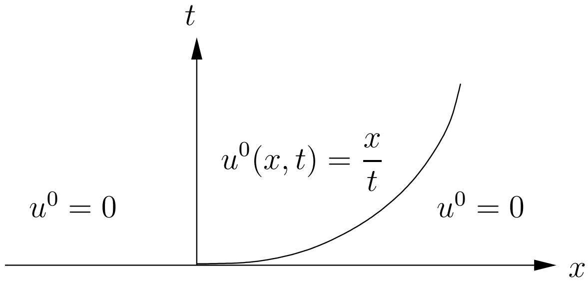


Figure 1.3. *The limit profile*

We observe that:

(*i*) $\|u^0(\cdot,t)\|_{L^\infty(\mathbb{R})} = \sqrt{\frac{2M}{t}}$ for $t > 0$.

(*ii*) supp $\big(u^0(\cdot,t)\big) = \overline{\{x \in \mathbb{R} : u^0(x,t) \neq 0\}} = [0, \sqrt{2Mt}\,]$.

(*iii*) speed of discontinuity (shock):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{M}{2t}} = \frac{u^0(x(t)-,t) + u^0(x(t)+,t)}{2} \; .$$

Remark 1.2. *Note that the expression for the shock speed is identical to the one found in the travelling wave case. This is not a coincidence. It will be explained in Chapter 2.*

1.3 General initial value problem

Consider the problem

$$(\mathrm{B}) \begin{cases} u_t + \left(\dfrac{u^2}{2}\right)_x = \nu u_{xx} & \text{in } Q \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R}\,, \end{cases}$$

where $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})^\star$ satisfies certain growth conditions at $\pm\infty$ which we leave unspecified for the moment. To solve this problem we introduce the *Cole–Hopf transformation* (see [15],[37]). First set

$$u = \psi_x \; ,$$

which gives

$$\psi_{xt} + \left(\frac{\psi_x^2}{2}\right)_x = \nu\psi_{xxx} \; ,$$

and after integration (setting the constant of integration zero)

$$\psi_t + \frac{\psi_x^2}{2} = \nu\psi_{xx} \; .$$

$^\star L^1_{\mathrm{loc}}(\mathbb{R}) := \big\{v : \mathbb{R} \to \mathbb{R} : \int_a^b |v(x)|dx < \infty$ for any $-\infty < a < b < \infty\big\}$. This is the span of locally integrable functions on $\mathbb{R}$.

Next set

$$\psi = -2\nu \log \varphi \qquad (\varphi > 0) \ .$$

Then

$$-2\nu \frac{\varphi_t}{\varphi} + \frac{4\nu^2}{2} \frac{\varphi_x^2}{\varphi^2} = \nu \left\{ -2\nu \frac{\varphi_{xx}}{\varphi} + 2\nu \frac{\varphi_x^2}{\varphi^2} \right\} ,$$

which leads to the transformed problem

$$\text{(T)} \begin{cases} \varphi_t = \nu \varphi_{xx} & \text{in } Q \ , \\ \varphi(\cdot, 0) = \varphi_0(\cdot) & \text{on } \mathbb{R} \ , \end{cases}$$

where

$$\varphi_0(x) = \exp\left\{ -\frac{\psi}{2\nu} \right\} = \exp\left\{ -\frac{1}{2\nu} \int_0^x u_0(s) \mathrm{d}s \right\} \qquad \text{for } x \in \mathbb{R} \ . \tag{1.10}$$

Theorem 1.3. *If $\varphi_0 \in C(\mathbb{R})$ satisfies for some constant $c > 0$ the growth condition $\varphi_0(x) = \mathcal{O}\left(\exp\left\{cx^2\right\}\right)$ as $x \to \pm\infty$, then* (T) *has a unique classical solution φ given by*

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{\mathbb{R}} \varphi_0(s) \exp\left\{ -\frac{(x-s)^2}{4\nu t} \right\} \mathrm{d}s \ .$$

Proof. See FRIEDMAN [25]. □

As an immediate consequence we have

Theorem 1.4. *Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ be such that φ_0, given by* (1.10)*, satisfies the condition of Theorem 1.3. Then* (B) *has a unique solution $u \in C^\infty(Q)$ and*

$$u = \psi_x = -2\nu \frac{\varphi_x}{\varphi} = \frac{\displaystyle\int_{\mathbb{R}} \left(\frac{x-s}{t} \right) \exp\left\{ -\frac{G}{2\nu} \right\} \mathrm{d}s}{\displaystyle\int_{\mathbb{R}} \exp\left\{ -\frac{G}{2\nu} \right\} \mathrm{d}s} ,$$

where

$$G = G(s,x,t) = \int_0^s u_0(p) \mathrm{d}p + \frac{(x-s)^2}{2t} \ .$$

Application : stability of travelling waves.
Let

$$u_0(x) = \begin{cases} u_\mathrm{l} & \text{for } x < 0 \ , \\ u_\mathrm{r} & \text{for } x > 0 \ , \end{cases}$$

where $-\infty < u_\mathrm{r} < u_\mathrm{l} < \infty$. Then

$$G(s,x,t) = \begin{cases} u_\mathrm{l} s + \dfrac{(x-s)^2}{2t} & \text{for } s < 0 \ , \\[2ex] u_\mathrm{r} s + \dfrac{(x-s)^2}{2t} & \text{for } s > 0 \ . \end{cases}$$

Substitution gives (non-trivial exercise)

$$u(x,t) = u_{\mathrm{r}} + \frac{u_{\mathrm{l}} - u_{\mathrm{r}}}{1 + h(x,t) \exp\left\{ \dfrac{u_{\mathrm{l}} - u_{\mathrm{r}}}{2\nu}(x - ct) \right\}} , \tag{1.11}$$

where

$$c = \tfrac{1}{2}(u_{\mathrm{l}} + u_{\mathrm{r}}) ,$$

and

$$h(x,t) = \left\{ \int_{-\frac{(x-u_{\mathrm{r}}t)}{\sqrt{4\nu t}}}^{\infty} \exp\left\{-\xi^2\right\} \mathrm{d}\xi \right\} \bigg/ \left\{ \int_{\frac{(x-u_{\mathrm{l}}t)}{\sqrt{4\nu t}}}^{\infty} \exp\left\{-\xi^2\right\} \mathrm{d}\xi \right\} .$$

From this expression we observe

(*i*) $h(x,t) \to 0$ as $t \to \infty$, for fixed $x/t < u_{\mathrm{r}}$.

(*ii*) $h(x,t) \to 1$ as $t \to \infty$, for fixed $u_{\mathrm{r}} < x/t < u_{\mathrm{l}}$.

(*iii*) $h(x,t) \to \infty$ as $t \to \infty$, for fixed $x/t > u_{\mathrm{l}}$.

Therefore $u(x,t) \to f(x - ct)$ in the above sense, see Figure 1.4.

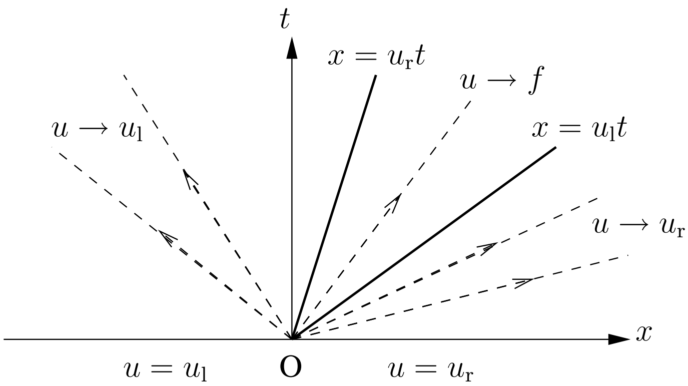


Figure 1.4. *Travelling wave limit*

Remark 1.5. *If for some* $a \neq 0$,

$$u_0(x) = \begin{cases} u_{\mathrm{l}} & \text{for } x < a , \\ u_{\mathrm{r}} & \text{for } x > a , \end{cases}$$

then u *converges to the shifted travelling wave:* $u(x,t) \to f(x - a - ct)$ *in the above sense.*

Remark 1.6. BURGERS [13] *wrote an interesting book on his equation. It contains a detailed discussion on the physical background and it gives an interpretation of solutions related to turbulent behaviour of flows. The book is called "The Nonlinear Diffusion Equation". This is remarkable in view of the Cole–Hopf transformation: the Burgers equation is one of the exceptional cases in which a nonlinear partial differential equation can be transformed to a linear one.*

2 The equation $u_t+(f(u))_x=0$

2.1 Characteristics

Throughout this chapter we assume $f\in C^2(\mathbb{R})$ and $f''>0$. We consider the initial value problem

$$(\mathrm{P})\begin{cases} u_t+\big(f(u)\big)_x=0 & \text{in } Q\ , \\ u(\cdot,0)=u_0(\cdot) & \text{on } \mathbb{R}\ . \end{cases} \tag{2.1}$$

Proposition 2.1. *The only smooth C^1-functions which satisfy equation* (2.1) *in Q are those which are non-decreasing in x for each fixed $t>0$.*

Proof. Let $u\in C^1(Q)$ be a solution of (2.1). Consider a point $(x_0,t_0)\in Q$ and the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}=f'\left(u(x,t)\right) & \text{for } t>0 \\ x(t_0)=x_0\ . \end{cases}$$

The unique solution $x(t)$ is a *characteristic curve* of equation (2.1). Along this curve we have

$$\frac{\mathrm{d}}{\mathrm{d}t}u\left(x(t),t\right)=u_t+u_x\frac{\mathrm{d}x}{\mathrm{d}t}=u_t+u_xf'(u)=0\ .$$

Thus u is constant along a characteristic. Consequently, the speed of a characteristic is also constant:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=f'\big(u(x_0,t_0)\big)\qquad \text{for } t>0\ .$$

Note that characteristics are straight lines in the $x-t$ plane. Now suppose there exist points (x_1,t_1), (x_2,t_1) $(x_2>x_1)$ such that

$$u_1:=u(x_1,t_1)>u(x_2,t_1)=:u_2\ .$$

Then

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}(t)=f'(u_1)>f'(u_2)=\frac{\mathrm{d}x_2}{\mathrm{d}t}(t)\qquad \text{for all } t>0\ .$$

Hence the characteristics intersect at some $t_2>t_1$, see Figure 2.1, contradicting the smoothness of u. $\square$

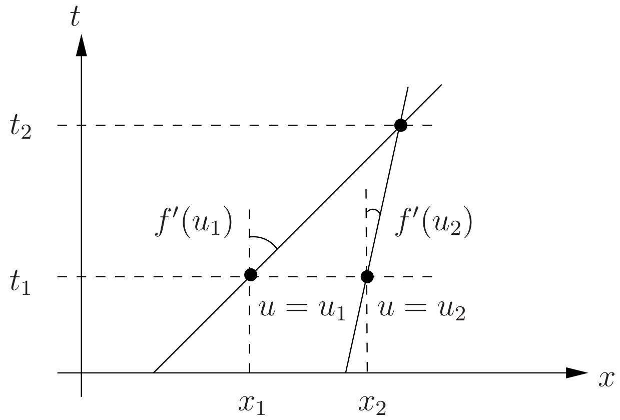


Figure 2.1. *Intersecting characteristics*

Example 2.2. *Let* $f(u)=\frac{1}{2}u^2$ *and*

$$u_0(x) = \begin{cases} 1 & x \leqslant 0 \ , \\ 1-x & 0 < x < 1 \ , \\ 0 & 1 \leqslant x \ . \end{cases}$$

Then, as in Figure 2.2,

$$u(x,t) = \begin{cases} 1 & x < t \\ (1-x)/(1-t) & t \leqslant x \leqslant 1 \\ 0 & x > 1 \ . \end{cases}$$

Note that the solution breaks down at $t=1$*!*

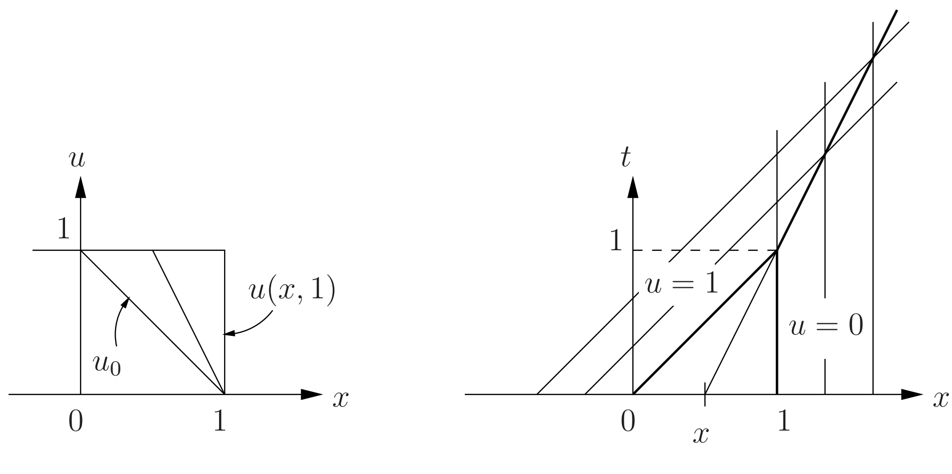


Figure 2.2. *Solution of example 2.2*

2.2 Construction by the method of characteristics

We recall that

- u is constant along any characteristic $x(t)$;
- $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f'\big(u(x,t)\big)$ for $t > 0$.

Let $(x,t) \in Q$ be a given point. Set

$$\begin{cases} u = u_0(y) \\ x - y = f'(u)t \quad \Rightarrow \quad y = x - f'(u)t \ . \end{cases}$$

Then $u = u(x,t)$ is implicitly given by the equation

$$u = u_0\big(x - f'(u)\,t\big) \qquad \text{for } (x,t) \in Q \ ,$$

see also Figure 2.3 for an explanation of the construction.

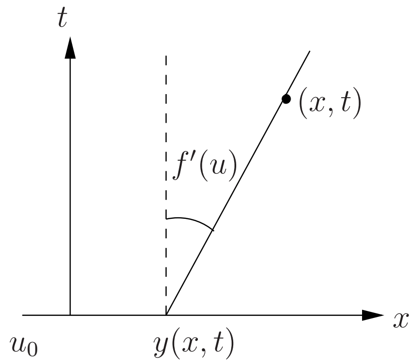


Figure 2.3. *The method of characteristics*

If $u_0 \in C^1(\mathbb{R})$ with u_0 and u_0' bounded on $\mathbb{R}$, we use the implicit function theorem to solve this equation for u as a differentiable function of x and t (with t sufficiently small). In particular

$$u_t = u_0' \left\{ -f''(u)u_t t - f'(u) \right\} \qquad \Rightarrow \qquad u_t = -\frac{f'(u)u_0'}{1 + f''(u)u_0' t}$$

and

$$u_x = u_0' \left\{ 1 - f''(u)u_x t \right\} \qquad \Rightarrow \qquad u_x = \frac{u_0'}{1 + f''(u)u_0' t} \ .$$

From these expressions we learn that if $f''(u)u_0' \geqslant 0$ (see also Proposition 2.1), then u_t and u_x remain bounded: the characteristics diverge and no discontinuity occurs. On the other hand, if $f''(u)u_0' < 0$, then the derivatives blow up when $1 + f''(u)u_0' t \to 0$. Since $f'' > 0$, this occurs if there are points where $u_0' < 0$. What happens near such a discontinuity or shock?

2.3 Weak solutions and shocks

Let u denote a density and f the corresponding mass flux. Further, let $f=f(u)$ be a given constitutive relation.

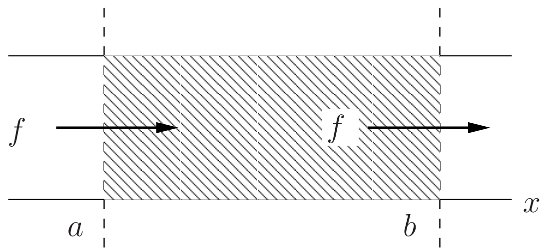


Figure 2.4. *Mass balance*

This leads to the mass-balance equation, see Figure 2.4,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x + \{f\left(u(b,t)\right) - f\left(u(a,t)\right)\} = 0\ ,$$

for any $-\infty<a<b<\infty$ and $t>0$. If u and f are smooth this gives

$$\int_a^b \left\{u_t + \left(f(u)\right)_x\right\} \mathrm{d}x = 0$$

and since this holds for any $a<b$ it follows that

$$u_t + \left(f(u)\right)_x = 0\ .$$

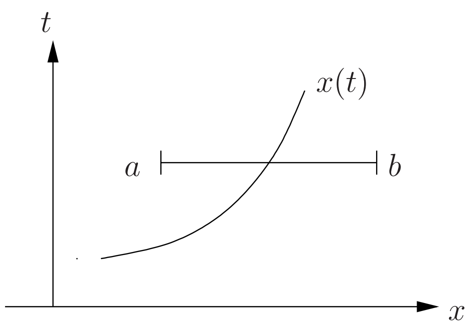


Figure 2.5. *Curve of discontinuity*

Next suppose that u is discontinuous across a smooth curve $x(t)$ and that the differential equation holds on both sides, see Figure 2.5. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_a^{x(t)} u(x,t)\mathrm{d}x + \int_{x(t)}^b u(x,t)\mathrm{d}x\right\}\\ &= \int_a^{x(t)} u_t\mathrm{d}x + \frac{\mathrm{d}x}{\mathrm{d}t}u(x(t)-,t) + \int_{x(t)}^b u_t\mathrm{d}x - \frac{\mathrm{d}x}{\mathrm{d}t}u(x(t)+,t)\\ &= \int_a^{x(t)} u_t\mathrm{d}x + \int_{x(t)}^b u_t\mathrm{d}x + \frac{\mathrm{d}x}{\mathrm{d}t}(u_\mathrm{l} - u_\mathrm{r})\\ &= f_\mathrm{a} - f_\mathrm{l} - f_\mathrm{b} + f_\mathrm{r} + \frac{\mathrm{d}x}{\mathrm{d}t}(u_\mathrm{l} - u_\mathrm{r})\ ,\end{aligned}$$

where indices are used to abbreviate notation. Thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f_\mathrm{r} - f_\mathrm{l}}{u_\mathrm{r} - u_\mathrm{l}} =: \frac{[f]}{[u]}\ . \tag{2.2}$$

This is called the *Rankine–Hugoniot shock condition.* It is a direct consequence of the conservation principle across the shock.

Finish example 2.2. *At $t = 1$ we have*

$$u(x,1) = \begin{cases} 1 & \textit{for } x < 1 \\ 0 & \textit{for } x > 1\ . \end{cases}$$

Continue the solution as a shock solution with $u_\mathrm{l} = 1$ and $u_\mathrm{r} = 0$. Then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{[f]}{[u]} = \frac{1}{2}\frac{u_\mathrm{r}^2 - u_\mathrm{l}^2}{u_\mathrm{r} - u_\mathrm{l}} = \frac{u_\mathrm{r} + u_\mathrm{l}}{2} = \frac{1}{2}\ .$$

Thus for $t \geqslant 1$ we may define

$$u(x,t) = \begin{cases} 1 & \textit{for } x < 1 + \frac{1}{2}(t-1) \\ 0 & \textit{for } x > 1 + \frac{1}{2}(t-1)\ . \end{cases}$$

Note that the shockspeed (2.2) *was also found for the limit cases discussed in Sections 1.1 and 1.2: with $f(u) = \frac{1}{2}u^2$, we have $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}(u_\mathrm{l} + u_\mathrm{r})$.*

Mathematically, the differential equation and the shock condition can be combined into one statement: the weak form of the differential equation.

Definition 2.3. *A bounded measurable function u is called a weak solution of* (P)*, with bounded measurable initial data u_0, if*

$$\int_Q \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \varphi(x,0)u_0(x)\mathrm{d}x = 0 \tag{2.3}$$

for all $\varphi \in C^1(\overline{Q})$ which vanish identically for large t and large $|x|$.

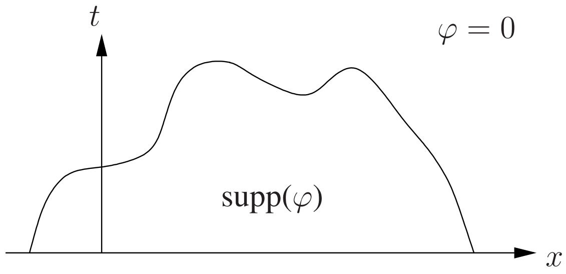


Figure 2.6. *Support of a typical test function* φ

Proposition 2.4. *Let* u *be a weak solution of* (P) *such that* $u \in C^1(N)$ *for some open set* $N \subset Q$*. Then equation* (2.1) *holds classically in* N.

Proof. For arbitrary $(x_0,t_0) \in N$, let $D \subset N$ denote a disc centered at (x_0,t_0). Let $\varphi \in C_0^\infty(D)$. Then identity (2.3) gives

$$\int_D \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t = 0$$

and the smoothness of u allows us to integrate by parts

$$\int_D \{u_t + (f(u))_x\}\,\varphi\mathrm{d}x\mathrm{d}t = 0 \qquad \text{for all } \varphi \in C_0^\infty(D)\ .$$

This implies that $u_t+(f(u))_x=0$ in (x_0,t_0). □

Proposition 2.5. *Let* $u \in C^1(\mathbb{R} \times (0,\delta)) \cap C(\mathbb{R} \times [0,\delta))$ *for some* $\delta > 0$*, and let* $u_0 \in C(\mathbb{R})$*. Then* $u(\cdot,0)=u_0(\cdot)$ *on* $\mathbb{R}$.

Proof. Let $\varphi \in C^1(\overline{Q})$ such that $\varphi(x,t)=0$ for $t \geqslant \delta$ and for large $|x|$. Integration by parts of (2.3) gives

$$-\int_{\mathbb{R}\times(0,\delta)} \{u_t + (f(u))_x\}\,\varphi\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \{u_0(x) - u(x,0)\}\,\varphi(x,0)\mathrm{d}x = 0\ .$$

The first term is zero. The second term implies $u_0(\cdot)=u(\cdot,0)$ on $\mathbb{R}$. □

Thus the definition of a weak solution is a true generalization of the classical concept of a solution.

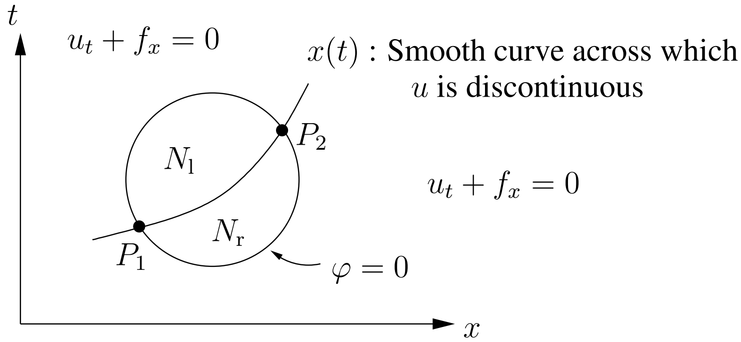


Figure 2.7. *Discontinuities*

What about discontinuities? Suppose we have a situation as in Figure 2.7. Let $\varphi \in C_0^\infty(N)$, $N \subset Q$. From (2.3) we have

$$0 = \int_N \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t = \int_{N_\mathrm{l}} \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t + \int_{N_\mathrm{r}} \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t\;.$$

Moreover

$$\begin{aligned}\int_{N_\mathrm{l}} \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t &= \int_{N_\mathrm{l}} \{(u\varphi)_t + (f(u)\varphi)_x\}\,\mathrm{d}x\mathrm{d}t \\ &= \int_{\partial N_\mathrm{l}} \varphi\,\{-u\mathrm{d}x + f(u)\mathrm{d}t\} = \int_{P_1}^{P_2} \varphi\,\{-u_\mathrm{l}\mathrm{d}x + f(u_\mathrm{l})\mathrm{d}t\}\;,\end{aligned}$$

and

$$\int_{N_\mathrm{r}} \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t = -\int_{P_1}^{P_2} \varphi\,\{-u_\mathrm{r}\mathrm{d}x + f(u_\mathrm{r})\mathrm{d}t\}\;,$$

where $u_\mathrm{l} = u(x(t)-,t)$ and $u_\mathrm{r} = u(x(t)+,t)$. Therefore

$$0 = \int_{P_1}^{P_2} \varphi\{[u]\mathrm{d}x - [f(u)]\mathrm{d}t\} \qquad \text{for all } \varphi \in C_0^\infty(N)\;,$$

which implies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{[f(u)]}{[u]} = \frac{f(u_\mathrm{r}) - f(u_\mathrm{l})}{u_\mathrm{r} - u_\mathrm{l}}\;.$$

Remark 2.6. *The conservation law has a divergent structure. Let*

$$\mathrm{div} := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) \qquad \text{and} \qquad \mathbf{q} := \big(f(u), u\big)\;.$$

The integral identity (2.3) *gives*

$$\mathrm{div}\,\mathbf{q} = 0 \qquad \text{in } D'(Q)\;,$$

by which we mean

$$\int_Q \mathbf{q}\cdot\mathrm{grad}\varphi\,\mathrm{d}x\mathrm{d}t = 0 \qquad \textit{for all } \varphi \in C_0^\infty(Q)\;.$$

This expression is well-defined because $\mathbf{q} \in (L^\infty(Q))^2$*. At points where* $\mathbf{q}$ *is smooth we have* $\mathrm{div}\,\mathbf{q} = 0$ *(classically), across smooth curves where* $\mathbf{q}$ *is discontinuous we have*

$$(\mathbf{q}_\mathrm{r} - \mathbf{q}_\mathrm{l})\cdot\mathbf{n} = 0\;,$$

where $\mathbf{n}$ *is the normal unit vector (see Figure 2.8)*

$$\mathbf{n} = -\cos\alpha\,\mathbf{e}_x + \sin\alpha\,\mathbf{e}_t\;.$$

This gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tan\alpha = \frac{(\mathbf{q}_\mathrm{r} - \mathbf{q}_\mathrm{l})_x}{(\mathbf{q}_\mathrm{r} - \mathbf{q}_\mathrm{l})_t} = \frac{[f(u)]}{[u]}\;.$$

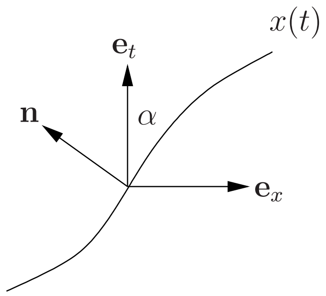


Figure 2.8. *Normal unit vector along curve*

The concept of a weak solution clearly unifies the differential equation and the Rankine–Hugoniot shock condition. The question we now pose relates to uniqueness. Does Definition 2.3 imply uniqueness for a given initial condition u_0? The answer is negative as can be seen from the following example.

Example 2.7. *Consider the Burgers equation with $\nu = 0$,*

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \text{in } Q \ ,$$

with

$$u(x,0) = \begin{cases} 0 & \textit{for } x < 0 \ , \\ 1 & \textit{for } x > 0 \ . \end{cases}$$

Here the shock-condition is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{f(u_\mathrm{r}) - f(u_\mathrm{l})}{u_\mathrm{r} - u_\mathrm{l}} = \frac{u_\mathrm{r} + u_\mathrm{l}}{2} \ .$$

One easily constructs the following solutions, see also Figure 2.9,

$$u_1(x,t) = \begin{cases} 0 & \textit{for } x < t/2 \\ 1 & \textit{for } x > t/2 \ , \end{cases}$$

and

$$u_2(x,t) = \begin{cases} 0 & \textit{for } x \leqslant 0 \\ x/t & \textit{for } 0 < x < t \\ 1 & \textit{for } x \geqslant t \ . \end{cases}$$

Example 2.8. *Consider again*

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad \text{in } Q \ ,$$

now with

$$u(x,0) = \begin{cases} 1 & \textit{for } x < 0 \\ -1 & \textit{for } x > 0 \ . \end{cases}$$

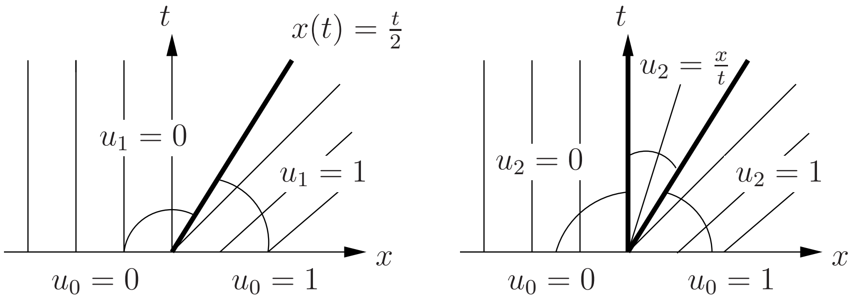


Figure 2.9. *Solutions of Example 2.7*

For each $\alpha \geqslant 1$, this problem has a solution u_α given by

$$u_\alpha(x,t) = \begin{cases} 1 & \textit{for } x < \dfrac{1-\alpha}{2}t\ , \\ -\alpha & \textit{for } \dfrac{1-\alpha}{2}t < x < 0\ , \\ +\alpha & \textit{for } 0 < x < \dfrac{\alpha-1}{2}t\ , \\ -1 & \textit{for } x > \dfrac{\alpha-1}{2}t\ . \end{cases}$$

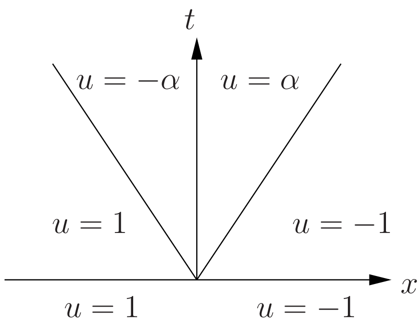


Figure 2.10. *Solutions of Example 2.8*

The solutions u_1 and u_α, with $\alpha > 1$, contain shocks for which $u_l < u_r$. From what we have learned in Section 1.1, these shocks cannot be obtained from a travelling wave type argument when the viscosity parameter ν tends to zero. We therefore expect that these solutions have no physical meaning.

Also from the point of view of characteristics such solutions are not likely to be physically meaningful, because characteristics on both sides of the shock diverge from the shock. Following LAX [45, 46] we now reject the solutions u_1 and u_α ($\alpha > 1$) for failure to satisfy the following criterion, see Figure 2.11:

> *Characteristics starting on either side of the shock curve, when continued in the direction of positive t, intersect this shock curve.*

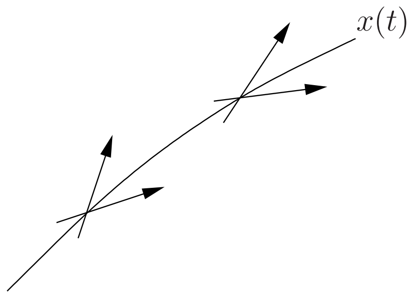


Figure 2.11. *Shock criterion of Lax*

This will be the case if

$$f'(u_l) > \frac{dx}{dt} = \frac{f(u_r) - f(u_l)}{u_r - u_l} > f'(u_r) \, .$$

Because $f'' > 0$ this means that

$$u_l > u_r \, . \tag{2.4}$$

Thus in addition to the Rankine–Hugoniot condition (2.2) we require that inequality (2.4) holds across a shock. For historical reasons we refer to condition (2.4) as the (Lax) *entropy condition.*

Condition (2.4) is a local condition at the shock. OLEINIK [56] replaced it by the following *global entropy condition*:

$$\frac{u(x+a,t) - u(x,t)}{a} \leqslant \frac{E}{t} \qquad \text{for all } a > 0 \text{ and for all } (x,t) \in Q \, , \tag{2.5}$$

where E is a positive constant independent of x, t and a.

Later on we show uniqueness for the initial value problem (P) within the class of weak solutions satisfying the entropy condition (2.5). We call the corresponding solution the *weak entropy solution.*

For convex flux functions f, inequality (2.5) captures the behaviour along characteristics as well as the Lax shock inequality (2.4). For smooth solutions we must have $u_0' \geqslant 0$ (see Proposition 2.1) and the method of characteristics gives

$$u_x = \frac{u_0'}{1 + f''(u)u_0' t} \, .$$

Thus if $u_0' = 0$, then $u_x = 0$ along the corresponding characteristic and if $u_0' > 0$ then

$$u_x < \frac{u_0'}{f''(u)u_0' t} = \frac{1}{f'' t} \leqslant \frac{E}{t} \qquad \text{with } E = \frac{1}{\inf f''} \, .$$

If a shock occurs at some $t > 0$, then (2.5) implies (taking a sufficiently small) that the solution can only jump downwards, giving $u_l > u_r$.

When dealing with discontinuous solutions, we have to be careful with the application of transformations.

Proposition 2.9. *Let u be a* smooth *solution of equation* (2.1). *Then $v := f'(u)$ satisfies the Burgers equation $v_t + \left(\frac{v^2}{2}\right)_x = 0$.*

Proof. Write the equation as

$$u_t + f'(u)u_x = 0\ .$$

Multiplication by f'' gives

$$f''(u)u_t + f'(u)f''(u)u_x = 0\ .$$

Then set $v = f'(u)$ and obtain

$$v_t + vv_x = 0 \qquad \text{in } Q\ .$$

□

Remark 2.10. *Proposition 2.9 does not hold in general for discontinuous solutions. This can be seen if we compare the shock speeds. The original equation gives*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{[f]}{[u]} = \frac{f(u_\mathrm{r}) - f(u_\mathrm{l})}{u_\mathrm{r} - u_\mathrm{l}}\ .$$

The transformed equation gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{[\frac{1}{2}v^2]}{[v]} = \frac{v_\mathrm{r} + v_\mathrm{l}}{2} = \frac{f'(u_\mathrm{r}) + f'(u_\mathrm{l})}{2}\ .$$

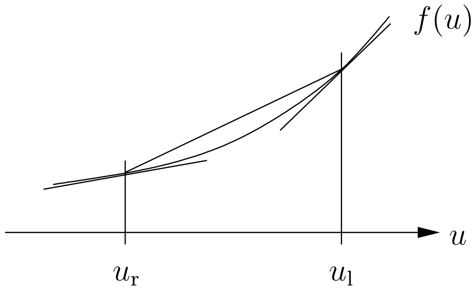


Figure 2.12. *Shock speeds for original and transformed equation*

Remark 2.11. *When $f \in C^2(\mathbb{R})$ and $f'' < 0$ (concave), the entropy condition becomes $u_\mathrm{l} < u_\mathrm{r}$; the only physically admissible shocks are those across which u increases.*

2.4 Rarefaction waves

Let $-\infty < u_\mathrm{l} < u_\mathrm{r} < \infty$ and consider the initial value problem (*Riemann problem*)

$$(\mathrm{R}) \begin{cases} u_t + \big(f(u)\big)_x = 0 & \text{in } Q\ , \\ u(x,0) = \begin{cases} u_\mathrm{l} & x < 0\ , \\ u_\mathrm{r} & x > 0\ . \end{cases} \end{cases}$$

We first give an intuitive argument. Let $u=u(x,t)$ denote the unique entropy solution of (R). Then for each $k>0$, the scaled functions

$$u_k(x,t) = u(kx,kt)$$

are also solutions of (R) satisfying the entropy condition. Then uniqueness gives

$$u(x,t) = u_k(x,t) = u(kx,kt) \qquad \text{for all } k>0 \text{ and for all } (x,t)\in Q\ .$$

Thus

$$u(x,\frac{1}{k}) = u(kx,1) \qquad \text{for all } k>0 \text{ and } x\in \mathbb{R}\ .$$

Consequently u must be of the form

$$u(x,t) = r(\eta) \qquad \text{with } \eta = \frac{x}{t}\ .$$

Formally this gives for r the equation

$$-\eta\frac{\mathrm{d}r}{\mathrm{d}\eta} + \frac{\mathrm{d}}{\mathrm{d}\eta}f(r) = \{-\eta + f'(r)\}\,\frac{\mathrm{d}r}{\mathrm{d}\eta} = 0 \qquad \text{in } \mathbb{R} \tag{2.6}$$

and the boundary conditions

$$r(-\infty) = u_\mathrm{l}\ , \qquad r(+\infty) = u_\mathrm{r}\ . \tag{2.7}$$

A solution of this boundary value problem (in an appropriate sense) is called a *rarefaction wave*.

Example 2.12. *Construction of rarefaction wave for the case* $0\leqslant u_\mathrm{l}<u_\mathrm{r}<\infty$ *and* $f(u)=u^p$ *for* $u\geqslant 0$ *with* $p>1$*. This leads to the boundary value problem*

$$\begin{cases} \{-\eta + p\,r^{p-1}\}\,\dfrac{\mathrm{d}r}{\mathrm{d}\eta} = 0 & \textit{in } \mathbb{R}\ ,\\ r(-\infty) = u_\mathrm{l}\ , \qquad r(+\infty) = u_\mathrm{r}\ . \end{cases}$$

We obtain as a solution

$$r(\eta) = \begin{cases} u_\mathrm{l} & \textit{for } \eta\leqslant\eta_\mathrm{l} := p u_\mathrm{l}^{p-1}\ ,\\ \left(\dfrac{\eta}{p}\right)^{1/(p-1)} & \textit{for } \eta_\mathrm{l}<\eta<\eta_\mathrm{r} := p u_\mathrm{r}^{p-1}\ ,\\ u_\mathrm{r} & \textit{for } \eta\geqslant\eta_\mathrm{r}\ . \end{cases}$$

Observe that

$$u(x,t) := \begin{cases} u_\mathrm{l} & \text{for } x<0,\quad t=0\ ,\\ r(x/t) & \text{for } (x,t)\in Q\ ,\\ u_\mathrm{r} & \text{for } x>0,\quad t=0\ , \end{cases}$$

satisfies

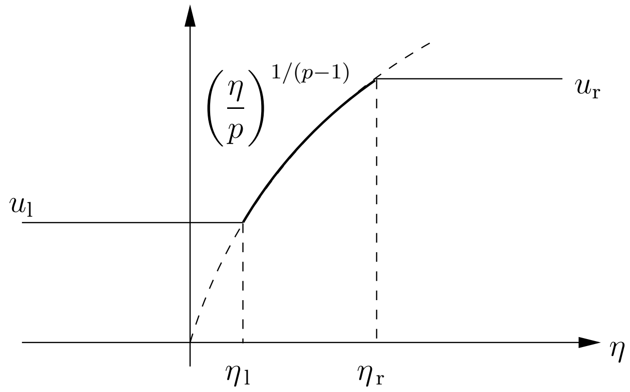


Figure 2.13. *Construction for $p > 1$*

(*i*) $u \in C\left(\overline{Q} \backslash O\right)$ and as in Figure 2.14:

$$u = \begin{cases} u_\mathrm{l} & \text{for } x \leqslant \eta_\mathrm{l} t\ , \\ u \in (u_\mathrm{l}, u_\mathrm{r}) \text{ and } u \text{ is } C^\infty & \text{for } \eta_\mathrm{l} t < x < \eta_\mathrm{r} t\ , \\ u_\mathrm{r} & \text{for } x \geqslant \eta_\mathrm{r} t\ ; \end{cases}$$

(*ii*) $u_t + (f(u))_x = 0$ in Q except when $x = \eta_\mathrm{l} t$ and $x = \eta_\mathrm{r} t$;

(*iii*) for $t > 0$ and $\eta_\mathrm{l} t < x < \eta_\mathrm{r} t$ we have

$$u_x(x,t) = \frac{1}{(p-1)\,p} \left(\frac{x}{pt}\right)^{\frac{2-p}{p-1}} \frac{1}{t}\ ;$$

This gives for $1 < p \leqslant 2$

$$u_x(x,t) \leqslant \frac{1}{(p-1)\,p} \left(\frac{\eta_\mathrm{r}}{p}\right)^{\frac{2-p}{p-1}} \frac{1}{t} = \frac{1}{(p-1)\,p} u_\mathrm{r}^{2-p} \frac{1}{t}\ ,$$

and for $p > 2$ and $u_\mathrm{l} > 0$

$$u_x(x,t) \leqslant \frac{1}{(p-1)\,p} \left(\frac{\eta_\mathrm{l}}{p}\right)^{\frac{2-p}{p-1}} \frac{1}{t} = \frac{1}{(p-1)\,p} u_\mathrm{l}^{2-p} \frac{1}{t}\ .$$

Note that the results of (*iii*) imply that the rarefaction wave satisfies the Oleinik entropy condition (2.5).

We want to derive the appropriate weak form for the boundary value problem (2.6), (2.7). This weak form allows us to consider equation (2.6) for a larger class of nonlinearities f. The starting point is the weak formulation for (R):

Find $u \in L^\infty(Q)$, $u_\mathrm{l} \leqslant u \leqslant u_\mathrm{r}$ a.e. in Q, such that

$$\int_\mathrm{Q} \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t + u_\mathrm{l} \int_{-\infty}^0 \varphi(x,0)\mathrm{d}x + u_\mathrm{r} \int_0^\infty \varphi(x,0)\mathrm{d}x = 0$$

for all admissible test functions.

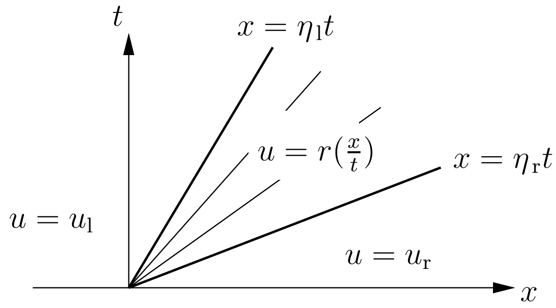


Figure 2.14. *Rarefaction wave*

Suppose this problem has a unique weak entropy solution u, which satisfies $u \in C(\overline{Q}\backslash O)$ and $u(x,0)=u_l$ for $x<0$ and $u(x,0)=u_r$ for $x>0$. Again by a simple scaling argument one finds that

$$u_k(x,t) := u(kx,kt)$$

is also a weak entropy solution for any $k>0$. As before this implies that the weak solution must be of the form

$$u(x,t)=r(\eta) \qquad \text{with } \eta=\frac{x}{t} \ ,$$

where $r\in L^\infty(\mathbb{R})\cap C(\mathbb{R})$. The initial condition and continuity imply that r satisfies the boundary conditions (2.7). In the integral identity we now choose test functions as in Figure 2.15: i.e.

$$\varphi(x,t)=\varphi_1(\eta)\cdot\varphi_2(t) \ ,$$

where $\varphi_1\in C_0^\infty(\mathbb{R})$ and $\varphi_2\in C_0^\infty(\mathbb{R}^+)$. For these test functions we have

$$\int_Q \{u\varphi_t+f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t=0 \ .$$

Since

$$\varphi_t=-\frac{\mathrm{d}\varphi_1}{\mathrm{d}\eta}\eta\frac{1}{t}\varphi_2+\varphi_1\frac{\mathrm{d}\varphi_2}{\mathrm{d}t}$$

and

$$\varphi_x=\frac{\mathrm{d}\varphi_1}{\mathrm{d}\eta}\frac{1}{t}\varphi_2 \ ,$$

we obtain

$$\int_0^\infty\left\{\int_{\mathbb{R}}\left(r\,(\eta)\left[t\frac{\mathrm{d}\varphi_2}{\mathrm{d}t}\varphi_1-\eta\frac{\mathrm{d}\varphi_1}{\mathrm{d}\eta}\varphi_2\right]+f\,(r)\,\frac{\mathrm{d}\varphi_1}{\mathrm{d}\eta}\varphi_2\right)\mathrm{d}\eta\right\}\mathrm{d}t=0$$

or

$$\int_0^\infty t\frac{\mathrm{d}\varphi_2}{\mathrm{d}t}\left(\int_{\mathbb{R}} r\varphi_1\mathrm{d}\eta\right)\mathrm{d}t+\int_0^\infty\varphi_2\left\{\int_{\mathbb{R}}(f\,(r)-\eta r)\,\frac{\mathrm{d}\varphi_1}{\mathrm{d}\eta}\mathrm{d}\eta\right\}\mathrm{d}t=0 \ .$$

The inner integrals (with respect to η) do not depend on t. We therefore can integrate the first term by parts. This leads to the following integral identity for r (dropping the index 1 for convenience):

$$\int_{\mathbb{R}}\left\{(f\,(r)-\eta r)\,\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}-r\varphi\right\}\mathrm{d}\eta=0 \qquad \text{for all } \varphi\in C_0^\infty(\mathbb{R}) \ . \tag{2.8}$$

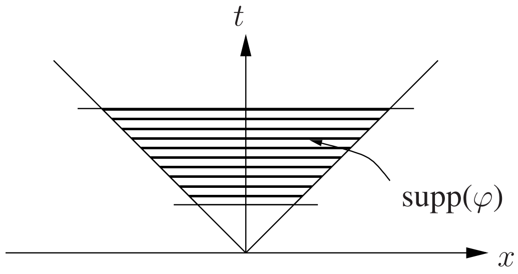


Figure 2.15. *The test function* $\varphi(x,t) = \varphi_1(\eta) \cdot \varphi_2(t)$

Definition 2.13. *(Weak formulation for rarefaction wave) A function* $r : \mathbb{R} \to \mathbb{R}$ *is called a rarefaction wave corresponding to boundary conditions* (2.7) *if*

(*i*) $r \in C\left((-\infty, \infty)\right)$

(*ii*) $r \in [u_\mathrm{l}, u_\mathrm{r}]$ and $r(-\infty) = u_\mathrm{l}$, $r(+\infty) = u_\mathrm{r}$

(*iii*) r satisfies identity (2.8).

Because $r \in C(\mathbb{R})$, identity (2.8) implies

$$f(r) - \eta r \in C^1(\mathbb{R}) \tag{2.9a}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{f(r) - \eta r\right\} + r = 0 \qquad \text{on } \mathbb{R} \tag{2.9b}$$

in the classical sense. To obtain (2.9b), we first write equation (2.8) as

$$\int_{\mathbb{R}} \left\{ f(r) - \eta r + \int_0^\eta r(s)ds \right\} \frac{\mathrm{d}\varphi}{\mathrm{d}\eta} \mathrm{d}\eta = 0$$

and use the following lemma.

Lemma 2.14. *Let* $I \subseteq \mathbb{R}$ *be an interval and let* $h \in L^1_{\mathrm{loc}}(I)$. *Suppose*

$$\int_I h \frac{\mathrm{d}\xi}{\mathrm{d}\eta} = 0 \qquad \textit{for all } \xi \in C_0^\infty(I)\ .$$

Then h *is constant a.e. in* I.

Proof. Let $\varphi, \varphi_1 \in C_0^\infty(I)$ such that $\int_I \varphi_1 = 1$. Set

$$\xi(\eta) = \int_{\partial I}^{\eta} \left\{ \varphi_1(x) \int_I \varphi - \varphi(x) \right\} \mathrm{d}x\ .$$

Then

$$\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \varphi_1 \int_I \varphi - \varphi$$

and

$$\int_I \left\{ h\varphi_1 \int_I \varphi \right\} = \int_I h\varphi \ .$$

Next let $c := \int_I h\varphi_1$. Then

$$\int_I c\varphi = \int_{\mathrm{I}} h\varphi \ , \qquad \text{for all } \varphi \in C_0^\infty(I) \ ,$$

which implies that $h = c$ a.e in I. □

Proposition 2.15. *Let* $f \in C^2((u_\mathrm{l}, u_\mathrm{r})) \cap C([u_\mathrm{l}, u_\mathrm{r}])$ *satisfy* $f'' > 0$. *Then there exists a rarefaction wave* r *of the form*

$$r(\eta) = \begin{cases} u_\mathrm{l} & \textit{for } \eta \leqslant \eta_\mathrm{l} \ , \\ (f')^{-1}(\eta) & \textit{for } \eta_\mathrm{l} < \eta < \eta_\mathrm{r} \ , \\ u_\mathrm{r} & \textit{for } \eta \geqslant \eta_\mathrm{r} \ , \end{cases} \tag{2.10}$$

where $\eta_\mathrm{l} := f'_\mathrm{r}(u_\mathrm{l}) \geqslant -\infty$ *and* $\eta_\mathrm{r} := f'_\mathrm{l}(u_\mathrm{r}) \leqslant +\infty$.

Proof. The assumptions on f imply that f' is C^1 and strictly increasing ($f'' > 0$) on $(u_\mathrm{l}, u_\mathrm{r})$. Consequently, the right limit at u_l and the left limit at u_r satisfy:

$$\lim_{u \downarrow u_\mathrm{l}} f'(u) = f'_\mathrm{r}(u_\mathrm{l}) \geqslant -\infty \qquad \text{and} \qquad \lim_{u \uparrow u_\mathrm{r}} f'(u) = f'_\mathrm{l}(u_\mathrm{r}) \leqslant +\infty \ ,$$

Let $A \subseteq \mathrm{I\!R}$ denote the range of f': i.e.

$$A = \{a \in \mathrm{I\!R} : \exists\, u \in (u_\mathrm{l}, u_\mathrm{r}) \text{ such that } f'(u) = a\} \ .$$

Clearly $A = (f'_\mathrm{r}(u_\mathrm{l}), f'_\mathrm{l}(u_\mathrm{r})) = (\eta_\mathrm{l}, \eta_\mathrm{r})$. We now define $r : (\eta_\mathrm{l}, \eta_\mathrm{r}) \to \mathrm{I\!R}$ by

$$\eta = f'(r(\eta)) \qquad \text{or} \qquad r(\eta) = (f')^{-1}(\eta) \qquad \text{for } \eta \in (\eta_\mathrm{l}, \eta_\mathrm{r}) \ .$$

Then $r \in (u_\mathrm{l}, u_\mathrm{r})$ and

$$\lim_{\eta \downarrow \eta_\mathrm{l}} r(\eta) = u_\mathrm{l} \qquad \text{and} \qquad \lim_{\eta \uparrow \eta_\mathrm{r}} r(\eta) = u_\mathrm{r} \ .$$

We extend this function by u_l for $\eta \leqslant \eta_\mathrm{l}$ (if $\eta_\mathrm{l} > -\infty$) and by u_r for $\eta \geqslant \eta_\mathrm{r}$ (if $\eta_\mathrm{r} < \infty$) and obtain (2.10). To show that indeed (2.10) is a rarefaction wave, we verify the conditions of Definition 2.13. From the construction we immediately see that (*i*) and (*ii*) are satisfied. To verify (*iii*) we show that r satisfies (2.9). The smoothness conditions on f imply

$$r \in C^1(\eta_\mathrm{l}, \eta_\mathrm{r}) \quad \text{with} \quad r'(\eta) = \frac{1}{f''(r(\eta))} \ , \qquad \text{for } \eta_\mathrm{l} < \eta < \eta_\mathrm{r} \ .$$

Hence (2.9) is satisfied on $(\eta_\mathrm{l}, \eta_\mathrm{r})$. Suppose $\eta_\mathrm{l} > -\infty$. Obviously (2.9) is also satisfied in $(-\infty, \eta_\mathrm{l})$. At $\eta = \eta_\mathrm{l}$ we have

$$(f(r) - \eta r)'(\eta_\mathrm{l}-) = -u_\mathrm{l}$$

and

$$(f(r) - \eta r)'(\eta_\mathrm{l}+) = \lim_{\eta \downarrow \eta_\mathrm{l}} \left\{ f'(r(\eta)) \frac{\mathrm{d}r}{\mathrm{d}\eta} - \eta \frac{\mathrm{d}r}{\mathrm{d}\eta} - r(\eta) \right\} = -u_\mathrm{l} \ .$$

Thus $f(r) - \eta r$ is differentiable at η_l and satisfies $(2.9b)$. A similar result holds at η_r. □

Next we drop the smoothness condition on f. We assume below that $f:[u_{\mathrm{l}},u_{\mathrm{r}}]\to\mathbb{R}$ is strictly convex and continuous up to the endpoints. For this case we have to revise expression (2.10). Before we do this we first give some properties of f:

(*i*) f strictly convex on $[u_{\mathrm{l}},u_{\mathrm{r}}]$ means that for all $u,v\in[u_{\mathrm{l}},u_{\mathrm{r}}]$ and for all $\lambda\in(0,1)$

$$f(\lambda u+(1-\lambda)v)<\lambda f(u)+(1-\lambda)f(v)\ ;$$

(*ii*) $f\in C([u_{\mathrm{l}},u_{\mathrm{r}}])$;

(*iii*) for any $u\in[u_{\mathrm{l}},u_{\mathrm{r}})$, $f'_{\mathrm{r}}(u)$ exists ($f'_{\mathrm{r}}(u_{\mathrm{l}})=-\infty$ possible) and $f'_{\mathrm{r}}(\cdot)$ is strictly increasing;
for any $u\in(u_{\mathrm{l}},u_{\mathrm{r}}]$, $f'_{\mathrm{l}}(u)$ exists ($f'_{\mathrm{l}}(u_{\mathrm{r}})=+\infty$ possible) and $f'_{\mathrm{l}}(\cdot)$ is strictly increasing;
for any $u\in(u_{\mathrm{l}},u_{\mathrm{r}})$, $f'_{\mathrm{r}}(u)\geqslant f'_{\mathrm{l}}(u)$;
for any $u,v\in(u_{\mathrm{l}},u_{\mathrm{r}})$ with $u<v$, $f'_{\mathrm{r}}(u)<f'_{\mathrm{l}}(v)$.

We introduce the subgradient of f in a point $u\in(u_{\mathrm{l}},u_{\mathrm{r}})$ as the interval

$$\partial f(u):=\Big\{k\in\mathbb{R}: f'_{\mathrm{l}}(u)\leqslant k\leqslant f'_{\mathrm{r}}(u)\Big\}\ .$$

This leads to the subgradient of f on $(u_{\mathrm{l}},u_{\mathrm{r}})$ as the graph $\partial f(u)$ for $u_{\mathrm{l}}<u<u_{\mathrm{r}}$. Note that if f is differentiable in $u_0\in(u_{\mathrm{l}},u_{\mathrm{r}})$, then

$$\partial f(u_0)=f'(u_0)\ ,$$

i.e. the subgradient consists of one point only.

Example 2.16. *Let $f:\mathbb{R}\to\mathbb{R}$ be given by*

$$f(x)=|x|+\frac{x^2}{2}\qquad \text{for } x\in\mathbb{R}\ .$$

Then the subgradient is the graph

$$\partial f(x)=\begin{cases}-1+x & x<0\ ,\\ [-1,1] & x=0\ ,\\ 1+x & x>0\ .\end{cases}$$

The last property in (*iii*) implies that $\partial f(\cdot)$ is a strictly increasing graph. Let again $A\subseteq\mathbb{R}$ denote the range of ∂f: i.e.

$$A=\Big\{a\in\mathbb{R}:\ \exists\, u\in(u_{\mathrm{l}},u_{\mathrm{r}}) \text{ such that } \partial f(u)\ni a\Big\}\ .$$

Then again

$$A=\big(f'_{\mathrm{r}}(u_{\mathrm{l}}),(f'_{\mathrm{l}}(u_{\mathrm{r}})\big)=(\eta_{\mathrm{l}},\eta_{\mathrm{r}})\ .$$

We now define $r:(\eta_{\mathrm{l}},\eta_{\mathrm{r}})\to\mathbb{R}$ by

$$\eta\in\partial f(r(\eta))\qquad\text{or}\qquad r(\eta)=\partial f^{-1}(\eta)\ .$$

Then $r\in C((\eta_{\mathrm{l}},\eta_{\mathrm{r}}))$ and r is non-decreasing with $u_{\mathrm{l}}<r<u_{\mathrm{r}}$ on $(\eta_{\mathrm{l}},\eta_{\mathrm{r}})$. Further

$$\lim_{\eta\downarrow\eta_{\mathrm{l}}} r(\eta)=u_{\mathrm{l}}\qquad\text{and}\qquad \lim_{\eta\uparrow\eta_{\mathrm{r}}} r(\eta)=u_{\mathrm{r}}\ .$$

On the remaining intervals $(-\infty,\eta_{\mathrm{l}}]$ and $[\eta_{\mathrm{r}},\infty)$ (if they exist) we extend r by the constants u_{l} and u_{r}, respectively. We have

Proposition 2.17. *Let $f:[u_\mathrm{l},u_\mathrm{r}]\to\mathbb{R}$ be strictly convex and continuous up to the endpoints. Then there exists a rarefaction wave of the form*

$$r(\eta)=\begin{cases} u_\mathrm{l} & \textit{for } \eta\leqslant\eta_\mathrm{l}\,,\\ \partial f^{-1}(\eta) & \textit{for } \eta_\mathrm{l}<\eta<\eta_\mathrm{r}\,,\\ u_\mathrm{r} & \textit{for } \eta\geqslant\eta_\mathrm{r}\,. \end{cases} \tag{2.11}$$

Proof. We have to check that (2.11) is a rarefaction wave in the sense of Definition 2.13. This is done by regularizing f, applying Proposition 2.15 and passing to the limit. We omit the details. □

2.5 Irreversibility

We show here by means of an example that the backwards problem is ill-posed.

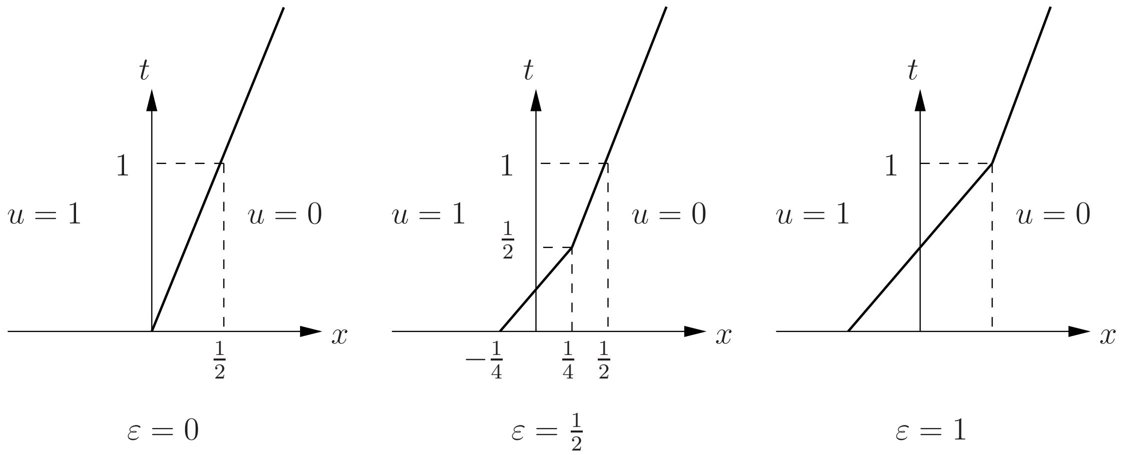


Figure 2.16. *Ill-posedness of the backwards problem*

Example 2.18. *Consider the equation $u_t+\left(\frac{u^2}{2}\right)_x=0$ in Q. For $0\leqslant\epsilon\leqslant 1$ we have the family of weak entropy solutions*

$$u_\epsilon(x,t)=\begin{cases} 1 & \textit{for } x\leqslant t-\epsilon/2\,,\\ (x-\epsilon/2)/(t-\epsilon) & \textit{for } t-\epsilon/2<x<\epsilon/2\,,\\ 0 & \textit{for } x\geqslant\epsilon/2\,, \end{cases} \qquad \text{for } t\leqslant\epsilon\,,$$

and

$$u_\epsilon(x,t)=\begin{cases} 1 & \textit{for } x<t/2\,,\\ 0 & \textit{for } x>t/2\,, \end{cases} \qquad \text{for } t>\epsilon\,.$$

All these solutions coincide identically for $t\geqslant 1$, although they emerge from different initial values. We show later that the solution operator defines a compact map from $L^\infty(\mathbb{R})$ into $L^1_{\mathrm{loc}}(\mathbb{R})$: i.e. if we denote the weak entropy solution of (P) *by $u(t;u_0)$, then $u(t;\cdot):L^\infty(\mathbb{R})\to L^1_{\mathrm{loc}}(\mathbb{R})$ is compact for all $t>0$. Hence the inverse cannot be continuous.*

3 Decay of the entropy solution

Here we consider the large time behaviour of solutions of the problem

$$\text{(P)} \begin{cases} u_t + (f(u))_x = 0 & \text{in } Q \ , \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R} \ . \end{cases} \tag{3.1}$$

Define the following constants

$$k := f''(0) \qquad \text{and} \qquad \mu := \min_{[-M,M]} f'' > 0 \tag{3.2}$$

and

$$M := \sup_{\mathbb{R}} |u_0| \ .$$

We suppose that

H$_1$: u_0 *is piecewise* C^1 *and piecewise monotone in* $\mathbb{R}$ *with compact support.*

H$_2$: $f \in C^2([-M,M])$ *with* $f''>0$.

Let u denote the weak entropy solution of (P). By regularity theory, see for instance SCHAEFFER [65], we may assume that u is piecewise C^1 in Q and that shock curves, parametrized as functions of t, are also C^1, except at points where they collide to form a new shock.

We start with some general observations.

1. Regularity implied by inequality (2.5). Since u, possibly redefined on a set of measure zero, fulfils

$$\frac{u(x+a,t)-u(x,t)}{a} \leqslant \frac{E}{t} \qquad \text{for } (x,t) \in Q \ ,$$

the function $v(x,t) := u(x,t) - \dfrac{E}{t}x$ satisfies for all $a>0$

$$v(x+a,t)-v(x,t) = u(x+a,t)-u(x,t) - \frac{E}{t}a \leqslant 0 \ .$$

Hence $v(\cdot,t)$ is non-increasing on $\mathbb{R}$, which implies that $u(\cdot,t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for all $t>0$. In other words, $u(\cdot,t)$ is locally of bounded variation. From this we deduce (*i*) $u(x+,t), u(x-,t)$ exist;

(*ii*) discontinuities in $u(\cdot,t)$ occur in the form of jumps;

(*iii*) jumps are countable;

(*iv*) $\mathrm{V}_{\mathbb{R}}\, u(\cdot,t) \geqslant \sum$ jumps.

Recall: $f:[a,b]\to\mathbb{R}$ has *bounded variation* if

$$\mathrm{V}_{a,b}\, f = \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty \, .$$

Here P denotes the partition $\{x_0, x_1, ..., x_n\}$ with $a = x_0 < x_1 < ... < x_n = b$. The supremum is taken with respect to all partitions of the interval $[a,b]$.

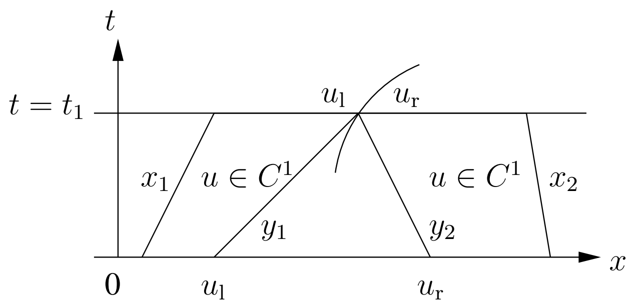


Figure 3.1. *Monotonicity of variation*

2. Suppose we have a situation as sketched in Figure 3.1. By the method of characteristics we have that

$$u(\cdot,0) \text{ on } (x_1(0), y_1(0)) \quad \text{and} \quad u(\cdot,t_1) \text{ on } (x_1(t_1), y_1(t_1))$$

are equivariant: i.e. the same values of u occur in the same order. This implies for the variations

$$V_{x_1(t_1),y_1(t_1)} u(\cdot,t_1) = V_{x_1(0),y_1(0)} u(\cdot,0) \qquad \text{for all } t_1 > 0 \, .$$

Similarly one has

$$V_{y_2(t_1),x_2(t_1)} u(\cdot,t_1) = V_{y_2(0),x_2(0)} u(\cdot,0) \qquad \text{for all } t_1 > 0 \, .$$

Since

$$V_{y_1(0),y_2(0)} u(\cdot,0) \geqslant u_l - u_r > 0 \, ,$$

we have

$$V_{x_1(t_1),x_2(t_1)} u(\cdot,t_1) \leqslant V_{x_1(0),x_2(0)} u(\cdot,0) \quad \text{for all } t_1 > 0 \, .$$

Hence the variation of $u(\cdot,t)$ is non-increasing with respect to t. This property gives the smoothing effect of the equation (or the characteristics).

3. For each $(x,t)\in Q$, the limits $u(x-,t)$ and $u(x+,t)$ exist. As shown in Figure 3.2, the characteristics through (x,t) with slopes $f'(u(x-,t))$ and $f'(u(x+,t))$ can be traced backwards in time up to $t=0$. This is due to the entropy condition. We conclude that $\sup_{\mathbb{R}}\, u(\cdot,t) \leqslant \sup_{\mathbb{R}}\, u_0$ for all $t>0$.

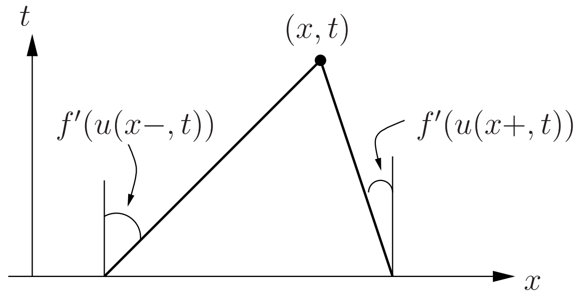


Figure 3.2. *Boundedness of solution*

4. For all $t>0$, $u(\cdot,t)$ has compact support. This follows from the maximum signal speed $f'(M)$ and the minimum signal speed $f'(-M)$, see Figure 3.3 and also the proof of Proposition 3.2.

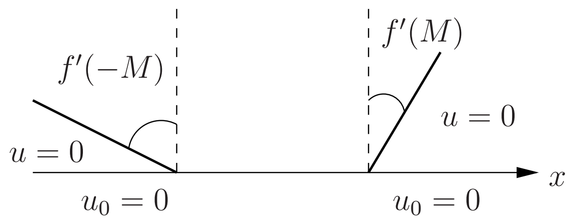


Figure 3.3. *Support of solution*

In the remainder of this chapter we study the boundary of the support and the large time behaviour of u.

Without loss of generality we may set

$$f(0) = f'(0) = 0 \ .$$

To see this we first transform according to:

$$\tilde{f}(u) = f(u) - f(0) - f'(0)u \ .$$

This gives

$$\begin{cases} u_t + (\tilde{f}(u))_x + f'(0)u_x = 0 & \text{in } Q \ , \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R} \ . \end{cases}$$

Then we set

$$v(x,t) = u(x + f'(0)t, t) \ ,$$

or

$$u(x,t) = v(x - f'(0)t, t) \ .$$

For v we find

$$\begin{cases} v_t + (\tilde{f}(v))_x = 0 & \text{in } Q \ , \\ v(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R} \ . \end{cases}$$

We introduce the following two numbers

$$q = \max_y \int_y^\infty u_0(x)\mathrm{d}x$$

and

$$-p = \min_y \int_{-\infty}^y u_0(x)\mathrm{d}x \; .$$

Clearly $p, q \geqslant 0$. For $t \geqslant 0$ we define

$$s^+(t) := \inf \{y : u(x,t) = 0 \text{ for all } x > y\} \; .$$

When $u_0 \not\equiv 0$ this is well-defined. Let $s_+ := s^+(0)$ and $u^+(t) := u(s^+(t)-, t)$. Obviously $u(s^+(t)+, t) \equiv 0$ for all $t > 0$. According to the entropy condition we must have $u^+(t) \geqslant 0$ for $t > 0$. In fact, we have

$$u^+(t) = 0 \text{ if and only if } x = s^+(t) \text{ is a point of continuity of } u(\cdot, t).$$

Proposition 3.1. *s^+ is non-decreasing in $\mathbb{R}^+$.*

Proof. We argue by contradiction. Let $0 \leqslant t_1 < t_2 < \infty$ and suppose $s^+(t_1) > s^+(t_2)$ as in Figure 3.4. Take any $x_0 \in (s^+(t_2), s^+(t_1))$. Clearly $u(x_0, t_2) = 0$. Then $f'(0) = 0$ implies that $u(x_0, t) \equiv 0$ for all $0 \leqslant t \leqslant t_2$. Since x_0 was chosen arbitrarily we also have

$$u(x, t_1) = 0 \quad \text{for all} \quad s^+(t_2) < x < s^+(t_1) \; ,$$

contradicting the definition of $s^+(t_1)$. □

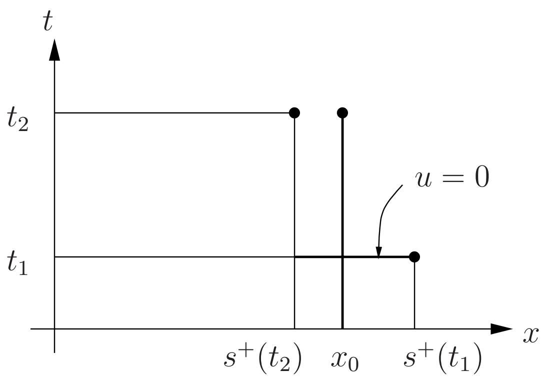


Figure 3.4. *Monotonicity of support (1)*

Proposition 3.2. *s^+ is uniformly Lipschitz continuous on $[0, \infty)$.*

Proof. For any $t_0 \geqslant 0$, consider the line

$$x - s^+(t_0) = f'(M)(t - t_0) \; .$$

Now suppose there exists a point (x^*, t^*) to the right of this line, with $t^* > t_0$, at which $u(x^*, t^*) \neq 0$. The characteristic passing through this point is also to the right of the line. This contradicts the definition of $s^+(t_0)$. □

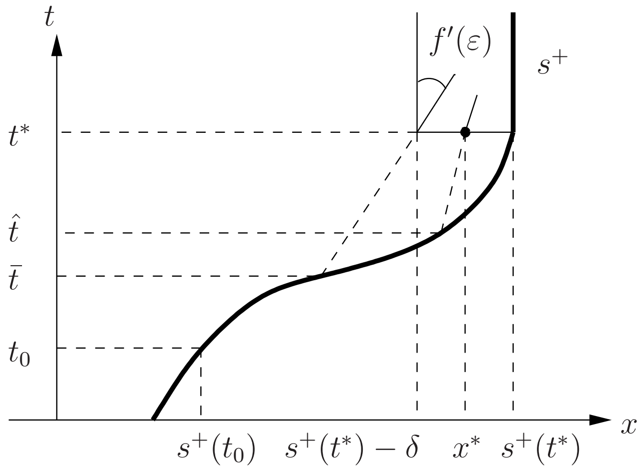


Figure 3.5. *Monotonicity of support (2)*

Proposition 3.3. *Suppose s^+ is constant in some interval $I \subset \mathbb{R}^+$. Then* $\inf I = 0$.

Proof. As in Figure 3.5 let us assume $t^* = \inf I > 0$. Then there exists a point $t_0 \in (0, t^*)$ such that $s^+(t_0) < s^+(t^*)$. Since s^+ is constant on I, it follows that $u^+(t^*) = 0$ and $x = s^+(t^*)$ is a point of continuity of $u(\cdot, t^*)$. Since u is piecewise C^1 (and hence piecewise defined) in Q, we have for given $\varepsilon > 0$ that there exists $\delta > 0$ such that

$$-\varepsilon < u(x, t^*) < \varepsilon \quad \text{for} \quad s^+(t^*) - \delta < x < s^+(t^*) \ .$$

Now choose ε and δ so that the line

$$x - (s^+(t^*) - \delta) = f'(\varepsilon)(t^* - t)$$

intersects s^+ in $\bar{t} \in (t_0, t^*)$. Let $x^* \in (s^+(t^*) - \delta, s^+(t^*))$, such that $u(x^*, t^*) \neq 0$. The characteristic going backwards from this point has to intersect s^+ in the interval $(\bar{t}, t^*)$, leading to a contradiction. Hence $t^* = 0$. ☐

We can now introduce the *waiting time*

$$T := \sup \left\{ t \geqslant 0 : s^+(t) = s_+ \right\} \ .$$

As a consequence of Proposition 3.3 we have

Corollary 3.4. *s^+ is strictly increasing in $[T, \infty)$.*

Next we estimate the growth of s^+. We first consider the trivial case

Proposition 3.5. *If $q = 0$ then $u^+(t) \equiv 0$ and $s^+(t) \equiv s_+$ for all $t > 0$.*

Proof. Suppose not. Then there exists a $\bar{t}$ such that $\bar{u} := u^+(\bar{t}) > 0$. Since $f'(\bar{u}) > 0$, a construction as in Figure 3.6 is possible. Since u is piecewise C^1 in Q, the divergence theorem can be applied. This gives

$$0 = \int_R (u_t + f(u)_x) =: \int_R \operatorname{div} \mathbf{q} = \int_{\partial R} \mathbf{q} \cdot \mathbf{n} \ . \tag{3.3}$$

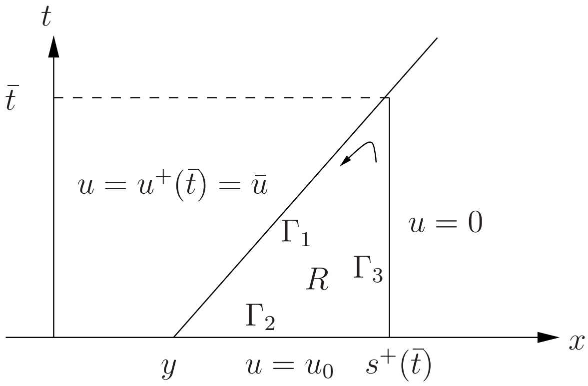


Figure 3.6. *Construction related to $q = 0$*

Using

$$\int_{\Gamma_1} \mathbf{q}\cdot\mathbf{n} = \Delta l(-f(\bar{u})\cos\alpha + \bar{u}\sin\alpha) = \Delta l \cos\alpha (f'(\bar{u})\bar{u} - f(\bar{u}))$$
$$= \bar{t}(f'(\bar{u})\bar{u} - f(\bar{u})) \ ,$$
$$\int_{\Gamma_2} \mathbf{q}\cdot\mathbf{n} = -\int_y^{s^+} u_0(x)\mathrm{d}x = -\int_y^{\infty} u_0(x)\mathrm{d}x$$

and

$$\int_{\Gamma_3} \mathbf{q}\cdot\mathbf{n} = 0 \ ,$$

we obtain

$$\bar{t}(f'(\bar{u})\bar{u} - f(\bar{u})) = \int_y^{\infty} u_0(x)\mathrm{d}x \ . \tag{3.4}$$

The convexity of f gives

$$\frac{f(\bar{u})}{\bar{u}} < f'(\bar{u}) \ ,$$

which implies

$$\int_y^{\infty} u_0(x)\mathrm{d}x > 0 \ .$$

This contradicts $q = 0$. Hence $\bar{u} = 0$. Since $f'(0) = 0$, the assertion for s^+ is immediate. □

More interesting is the case

Proposition 3.6. *If $q > 0$ then $s^+(t) \leqslant s_+ + c\left(\frac{2q}{\mu}\right)^{\frac{1}{2}} t^{\frac{1}{2}}$ for all $t \geqslant 0$, where $c = \max\limits_{[-M,M]} f''$.*

Proof. If $u^+(t)>0$ for some $t>0$, it follows from (3.4) that

$$f'(u^+)u^+ - f(u^+) \leqslant \frac{q}{t} \ . \tag{3.5}$$

Since for any $a\in\mathbb{R}$

$$0 = f(0) = f(a) + f'(a)(0-a) + \frac{1}{2}f''(\xi)a^2 \geqslant f(a) - af'(a) + \frac{1}{2}\mu a^2 \ , \tag{3.6}$$

we find

$$u^+(t) \leqslant \left(\frac{2q}{\mu}\right)^{\frac{1}{2}} t^{-\frac{1}{2}} \ .$$

Using

$$\dot{s}^+(t) = \frac{f(u^+(t))}{u^+(t)} = \frac{f''(\theta)}{2}u^+(t) \qquad \text{for } 0<\theta<u^+(t) \ , \tag{3.7}$$

we obtain

$$\dot{s}^+(t) \leqslant \frac{c}{2}\left(\frac{2q}{\mu}\right)^{\frac{1}{2}} t^{-\frac{1}{2}} \ .$$

Here $\dot{}$ means differentiation with respect to t. The estimate follows after integration. □

For the left boundary we find similar results. If we define for $t\geqslant 0$

$$s^-(t) := \sup\{y : u(x,t) = 0 \text{ for all } x<y\} \ ,$$

with $s_- = s^-(0)$ we obtain:

$$p = 0 \quad \Rightarrow \quad s^-(t) = s_- \qquad\qquad \text{for all } t\geqslant 0 \ ; \tag{3.8a}$$

$$p > 0 \quad \Rightarrow \quad s^-(t) \geqslant s_- - c\left(\frac{2p}{\mu}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \qquad \text{for all } t\geqslant 0 \ . \tag{3.8b}$$

The spreading of the support of u is sketched in Figure 3.7.

The decay of the solution follows from an argument with characteristics. For any (x_0,t_0), with $t_0>0$ and $s^-(t_0)<x_0<s^+(t_0)$, consider the two characteristics $x_\pm$, satisfying

$$x_\pm(t) = x_0 + f'\big(u(x_0\pm, t_0)\big)(t-t_0) \ .$$

They intersect the $x-$axis at $y_\pm$ satisfying

$$s_- \leqslant y_- \leqslant y_+ \leqslant s_+ \ .$$

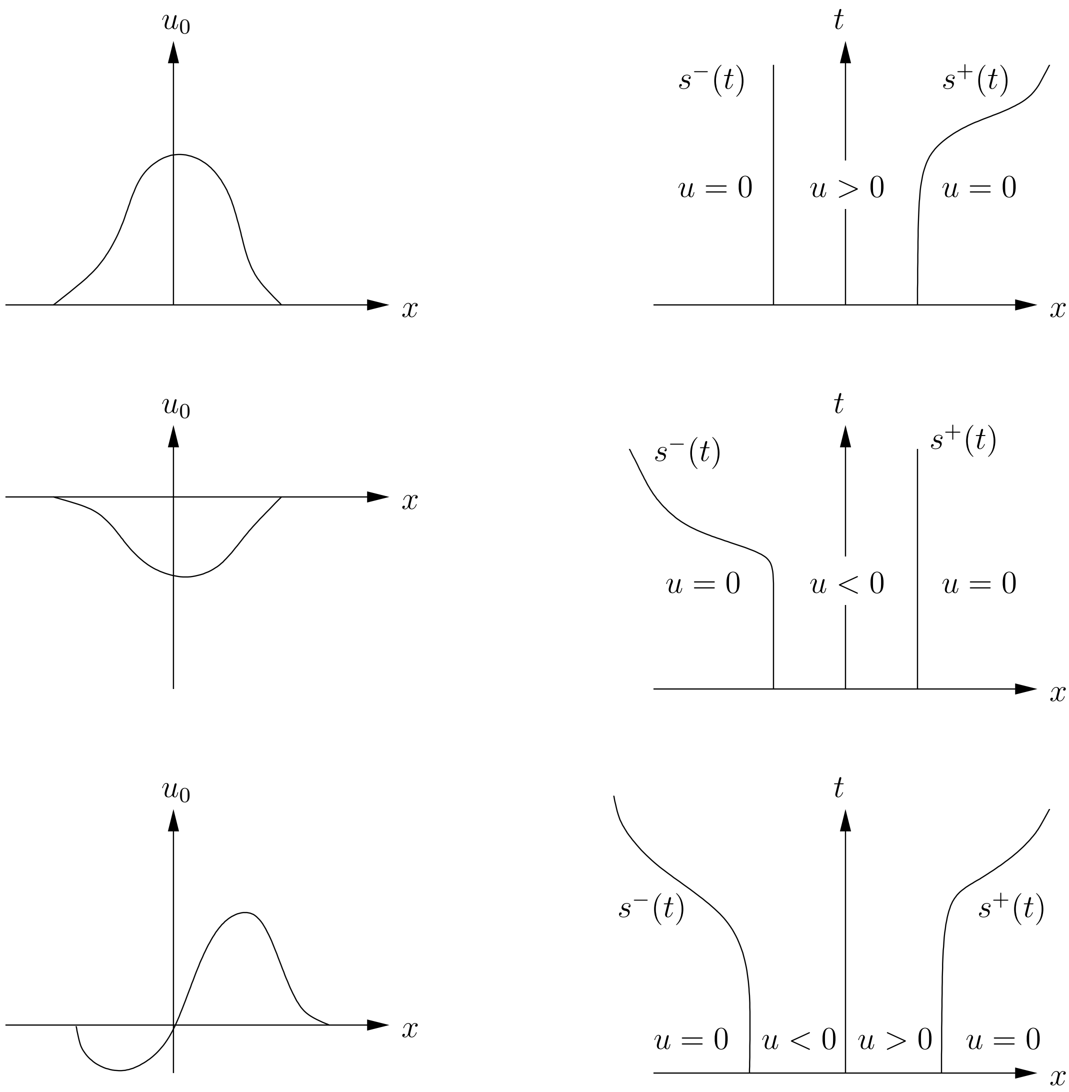


Figure 3.7. *Spreading of the support of u*

Thus

$$\begin{aligned}|f'\big(u(x_0\pm,t_0)\big)| = \left|\frac{y_\pm - x_0}{t_0}\right| &\leqslant \frac{1}{t_0}\max\big\{(s^+(t_0)-s_-),(s_+-s^-(t_0))\big\}\\ &\leqslant \frac{(s_+-s_-)+\text{Const }\ t_0^{\frac{1}{2}}}{t_0}\\ &\leqslant \text{Const }\ t_0^{-\frac{1}{2}}\end{aligned}$$

for t_0 sufficiently large. But $|f'(a)| = \left|\int_0^a f''(s)\mathrm{d}s\right| \geqslant \mu|a|$. Hence $|u(x_0,t_0)| \leqslant \text{Const }\ t_0^{-\frac{1}{2}}$. Thus we have shown

Theorem 3.7. *Let* H_1 *and* H_2 *be satisfied such that* $f(0) = f'(0) = 0$*. Then*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} = \mathcal{O}(t^{-\frac{1}{2}}) \qquad \text{as } t\to\infty\ .$$

Next we determine the *asymptotic profile*. For technical reasons we need $f \in C^3((-M,M))$. From (3.5)–(3.7) we obtain

$$\dot{s}^+(t) = \frac{f''(\theta)}{2} u^+(t) \leqslant \frac{1}{2} \frac{f''(\theta)}{\{f''(\xi)\}^{\frac{1}{2}}} \left(\frac{2q}{t}\right)^{\frac{1}{2}} ,$$

where $0 < \theta, \xi < u^+(t)$. Hence $\theta, \xi = \mathcal{O}(t^{-\frac{1}{2}})$. The smoothness of f gives

$$\dot{s}^+(t) \leqslant \left(\frac{qk}{2t}\right)^{\frac{1}{2}} + \mathcal{O}(t^{-1}) \qquad \text{for } t > 0 ,$$

from which we deduce

$$s^+(t) \leqslant s_+ + (2qkt)^{\frac{1}{2}} + \mathcal{O}(\ln t) \qquad \text{for } t > 0 .$$

Similarly

$$s^-(t) \geqslant s_- - (2pkt)^{\frac{1}{2}} + \mathcal{O}(\ln t) \qquad \text{for } t > 0 .$$

Next we define the functions

$$w(x,t) = \begin{cases} \dfrac{x}{kt} & \text{for } t > 0 \text{ and } s_- - (2pkt)^{\frac{1}{2}} < x < s_+ + (2qkt)^{\frac{1}{2}} \\ 0 & \text{elsewhere} \end{cases} \tag{3.9}$$

and

$$\tilde{w}(x,t) = \begin{cases} \dfrac{x}{kt} & \text{for } t > 0 \text{ and } s^-(t) < x < s^+(t) \\ 0 & \text{elsewhere} . \end{cases} \tag{3.10}$$

The goal is to find an estimate for $u - w$. Using again the method of characteristics we obtain for any $(x,t) \in Q$ with $t > 0$, $s^-(t) < x < s^+(t)$ and u continuous at (x,t), a unique $y = y(x,t) \in (s_-, s_+)$ such that

$$f'(u(x,t)) = \frac{x - y(x,t)}{t} .$$

This implies

$$f''(0)u(x,t) + \mathcal{O}(u^2) = \frac{x - y(x,t)}{t}$$

and thus

$$u(x,t) = \frac{x}{kt} + \mathcal{O}(t^{-1}) \qquad \text{for } t > 0 \text{ and } s^-(t) < x < s^+(t) .$$

At points where u is discontinuous we take the left or right limit in the above expressions. Hence for $t > 0$ and sufficiently large

$$\begin{aligned} \|u(\cdot,t) - \tilde{w}(\cdot,t)\|_{L^1(\mathbb{R})} := \int_{\mathbb{R}} |u(x,t) - \tilde{w}(x,t)| \mathrm{d}x &\leqslant \text{Const } t^{-1}(s^+(t) - s^-(t)) \\ &\leqslant \text{Const } t^{-1}(1 + t^{\frac{1}{2}} + \ln t) \\ &\leqslant \text{Const } t^{-\frac{1}{2}} . \end{aligned}$$

Similarly we obtain, again for t sufficiently large,

$$\|w(\cdot,t) - \tilde{w}(\cdot,t)\|_{L^1(\mathbb{R})} \leqslant \text{Const } t^{-\frac{1}{2}} .$$

Combining these estimates we have the convergence result

Theorem 3.8. *Suppose in addition $f \in C^3((-M,M))$. Then*

$$\|u(\cdot,t) - w(\cdot,t)\|_{L^1(\mathbb{R})} = \mathcal{O}(t^{-\frac{1}{2}}) \qquad \text{as } t \to \infty \ .$$

The function w is called a N-wave because of its profile at each fixed $t > 0$.

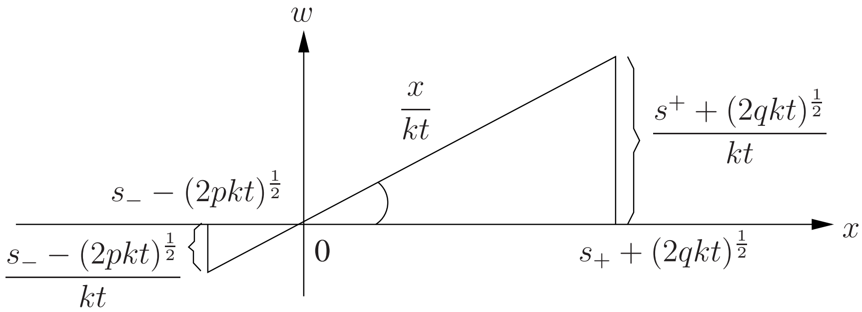


Figure 3.8. *N-wave*

Remark 3.9. *Suppose we take $s_+ = s_- = 0$ in the expression for w. Then*

$$\int_{\mathbb{R}} |w(x,t)| \mathrm{d}x = p + q \qquad \textit{for all } t > 0 \ . \tag{3.11}$$

This is consistent with the following observation. For $s_+ = s_- = 0$, w satisfies the equation

$$w_t + \left(\frac{1}{2} k w^2\right)_x = 0 \qquad \textit{in } Q \tag{3.12}$$

and

$$\lim_{t \downarrow 0} w(x,t) = -p\delta(0-) + q\delta(0+) \ . \tag{3.13}$$

Thus the N-wave is the dipole solution of the Burgers equation with $f(u) = \frac{1}{2} k u^2$. If u_0 is as in Figure 3.7 (bottom), it follows that

$$\int_{\mathbb{R}} |u(x,t)| \mathrm{d}x = p + q \qquad (p, q > 0) \tag{3.14}$$

as well. In that case the L^1-norm of u and w are the same for all $t > 0$ (or asymptotically the same if $s_- < 0 < s_+$). Theorem 3.8 tells us how u redistributes as t increases.

4 Uniqueness of the entropy solution

In this chapter we prove the uniqueness of the weak entropy solution of

$$(\mathrm{P})\begin{cases} u_t + \big(f(u)\big)_x = 0 & \text{in } Q \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R}\ . \end{cases}$$

We recall that a weak solution of this initial value problem satisfies

(*i*) $u \in L^\infty(Q)$;

(*ii*) $\int_Q \{u\varphi_t + f(u)\varphi_x\}\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\mathrm{d}x = 0$ for all $\varphi \in C^1(\overline{Q})$ such that φ vanishes identically for large $|x|$ and large t. We denote this class of test functions by $C_0^1(t \geqslant 0)$.

A weak *entropy* solution satisfies in addition (u possibly redefined on a set of measure zero)

$$\frac{u(x+a,t)-u(x,t)}{a} \leqslant \frac{E}{t}$$

for all $(x,t) \in Q$ and for all $a > 0$, where the constant E is positive and independent of x, t and a.

The uniqueness proof we present here is due to Oleinik [56], see also Smoller [67]. We first give the underlying idea.

Suppose u and v are two weak entropy solutions of (P), for the same initial value u_0. Then

$$\int_Q \{(u-v)\varphi_t + (f(u)-f(v))\,\varphi_x\}\,\mathrm{d}x\mathrm{d}t = 0 \qquad \text{for all } \varphi \in C_0^1(t \geqslant 0)\ ,$$

which we write as

$$\int_Q (u-v)\left\{\varphi_t + \frac{f(u)-f(v)}{u-v}\varphi_x\right\}\mathrm{d}x\mathrm{d}t = 0 \qquad \text{for all } \varphi \in C_0^1(t \geqslant 0)\ .$$

Here u and v are considered as known functions. Let

$$F(x,t) := \frac{f(u(x,t)) - f(v(x,t))}{u(x,t)-v(x,t)} \qquad \text{for } (x,t) \in Q\ .$$

Then we have

$$\int_Q (u-v)\{\varphi_t + F\varphi_x\}\,\mathrm{d}x\mathrm{d}t = 0 \qquad \text{for all } \varphi \in C_0^1(t \geqslant 0)\ .$$

Now suppose we could solve the *adjoint equation*

$$\varphi_t + F\varphi_x = \psi \in C_0^1(Q)$$

for $\varphi \in C_0^1(t \geqslant 0)$. Then we would have

$$\int_Q (u-v)\psi\,\mathrm{d}x\mathrm{d}t = 0 \qquad \text{for all } \psi \in C_0^1(Q)\ ,$$

implying $u=v$ a.e. in Q.

In general, however, u, v and thus F are not smooth and we cannot expect that the adjoint equation has a smooth solution for arbitrary ψ. The way to proceed is to approximate u and v (and thus F) by smooth functions $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ and then to solve the linear problem:

> *For* $n \in \mathbb{N}$*, find* $\varphi_n \in C_0^1(t \geqslant 0)$ *such that*
>
> $$(\varphi_n)_t + F_n(\varphi_n)_x = \psi \in C_0^1(Q) \qquad \text{in } Q\ ,$$
>
> *where*
>
> $$F_n = \frac{f(u_n) - f(v_n)}{u_n - v_n} \qquad \text{in } Q\ .$$

Assuming that this problem has a solution (existence question), we find

$$\begin{aligned}\int_Q (u-v)\psi &= \int_Q (u-v)(\varphi_n)_t + \int_Q (u-v)F_n(\varphi_n)_x \\ &= -\int_Q \{f(u)-f(v)\}(\varphi_n)_x + \int_Q (u-v)F_n(\varphi_n)_x \\ &= \int_Q (u-v)\{F_n - F\}(\varphi_n)_x\ .\end{aligned}$$

for all $n \in \mathbb{N}$. We will show that $F_n \to F$ locally in $L^1(\overline{Q})$ and that $(\varphi_n)_x$ and supp(φ_n) are bounded, uniformly in $n \in \mathbb{N}$. Then passing to the limit in the right hand side gives

$$\int_Q (u-v)\psi = 0\ ,$$

implying the uniqueness.

The convergence of $\{F_n\}_{n\in\mathbb{N}}$ is a direct consequence of the way we approximate the solutions u and v. To establish the uniform bound on $(\varphi_n)_x$ we use the entropy condition.

Theorem 4.1. *Suppose* u *and* v *are two weak entropy solutions of* (P)*, in which* $f \in C^2([-M,M])$ *and* $f''(s) > 0$ *for* $s \in [-M,M]$ *where* $M = \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$*.Then*

$$u = v \qquad \text{a.e. in } Q\ .$$

Proof. Given $u, v \in L^\infty(Q)$, we define the approximations by convolutions. First introduce *mollifiers* $\{\rho_n\}_{n\in\mathbb{N}}$, which satisfy

(*i*) $\rho_n \in C_0^\infty(\mathbb{R}^2)$;

(*ii*) $\rho_n \geqslant 0$;

(*iii*) $\displaystyle\int_{\mathbb{R}^2} \rho_n = 1$;

(*iv*) $\mathrm{supp}(\rho_n) \subseteq B_{1/n}(O)$.

Next we extend u and v to $\mathbb{R}^2$ by

$$\tilde{u} = \begin{cases} u & \text{in } Q\ , \\ 0 & \text{elsewhere}\ , \end{cases} \qquad \text{and} \qquad \tilde{v} = \begin{cases} v & \text{in } Q\ , \\ 0 & \text{elsewhere}\ , \end{cases}$$

and we consider the approximations

$$u_n(x,t) := \rho_n \star \tilde{u}(x,t) = \int_{\mathbb{R}^2} \rho_n(y-x, \tau-t)\tilde{u}(y,\tau)\mathrm{d}y\mathrm{d}t$$

and

$$v_n(x,t) := \rho_n \star \tilde{v}(x,t) = \int_{\mathbb{R}^2} \rho_n(y-x, \tau-t)\tilde{v}(y,\tau)\mathrm{d}y\mathrm{d}t$$

for $(x,t) \in \mathbb{R}^2$ and $n \in \mathbb{N}$. These approximations satisfy for all $n \in \mathbb{N}$

(*i*) $u_n, v_n \in C^\infty(\mathbb{R}^2)$;

(*ii*) $|u_n| \leqslant M$ and $|v_n| \leqslant M$ in $\mathbb{R}^2$;

(*iii*) if $\tilde{u} \in L^p(\mathbb{R}^2)$, $1 \leqslant p < \infty$, then $u_n \to \tilde{u}$ in $L^p(\mathbb{R}^2)$, i.e. $\|u_n - \tilde{u}\|_{L^p(\mathbb{R}^2)} \to 0$ as $n \to \infty$ (see ADAMS [2]);

(*iv*) if $u \in L^\infty(Q)$ then $u_n \to u$ in $L^1_{\mathrm{loc}}(\overline{Q})$ as $n \to \infty$. This means that for any $K \subset\subset \overline{Q}$, $\|u_n - u\|_{L^1(K)} \to 0$ as $n \to \infty$.

Property (*iv*) is a direct consequence of (*iii*) for $p = 1$. This we see as follows. Take an arbitrary $K \subset\subset \overline{Q}$ and consider the function

$$u^* = \begin{cases} u & \text{in } K \\ 0 & \text{in } \mathbb{R}^2 \backslash K\ . \end{cases}$$

Clearly $u^* \in L^1(\mathbb{R}^2)$ and $|u^*| \leqslant M$ in $\mathbb{R}^2$. Define $u_n^* := \rho_n \star u^*$. Then by (*iii*)

$$\|u_n^* - u^*\|_{L^1(K)} \leqslant \|u_n^* - u^*\|_{L^1(\mathbb{R})} \to 0 \qquad \text{as } n \to \infty\ .$$

Hence

$$\|u_n^* - u^*\|_{L^1(K)} \to 0 \qquad \text{as } n \to \infty \ .$$

Since $u_n = u_n^*$ in K_n (K_n is defined in Figure 4.1), we obtain from

$$u_n - u = u_n - u_n^* + u_n^* - u$$

that

$$\begin{aligned}\|u_n - u\|_{L^1(K)} &\leqslant \|u_n - u_n^*\|_{L^1(K)} + \|u_n^* - u^*\|_{L^1(K)} \\ &= \|u_n - u_n^*\|_{L^1(K/K_n)} + \|u_n^* - u^*\|_{L^1(K)} \ .\end{aligned}$$

Clearly the right hand side converges to zero for $n \to \infty$. This implies the result.

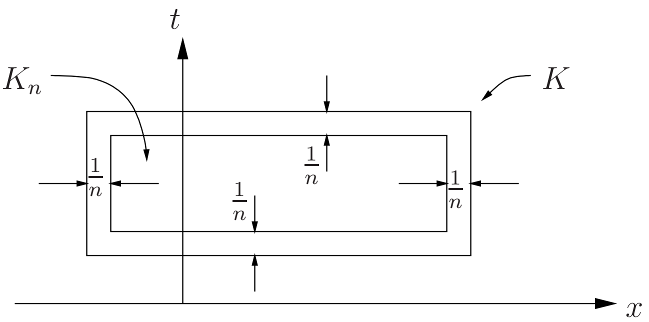


Figure 4.1. *Construction of rectangles K_n*

Using the approximations u_n and v_n we also define for $n \in \mathbb{N}$

$$F_n := \frac{f(u_n) - f(v_n)}{u_n - v_n} = \frac{1}{u_n - v_n} \int_{v_n}^{u_n} f'(s)\mathrm{d}s \qquad \text{in } Q \ .$$

We transform the integral according to

$$s = \vartheta u_n + (1-\vartheta)v_n \ , \qquad 0 \leqslant \vartheta \leqslant 1 \ ,$$

which gives

$$F_n = \int_0^1 f'\left(\vartheta u_n + (1-\vartheta)v_n\right)\mathrm{d}\vartheta \ .$$

From this expression we obtain

$$\frac{\partial F_n}{\partial x} = \int_0^1 f''\left(\vartheta u_n + (1-\vartheta)v_n\right)\left(\vartheta\frac{\partial u_n}{\partial x} + (1-\vartheta)\frac{\partial v_n}{\partial x}\right)\mathrm{d}\vartheta$$

and consequently

$$\left|\frac{\partial F_n}{\partial x}\right| \leqslant \frac{1}{2}\sup f''\left\{\left|\frac{\partial u_n}{\partial x}\right| + \left|\frac{\partial v_n}{\partial x}\right|\right\} .$$

A similar result holds for $\partial F_n/\partial t$. Hence F_n is continuous differentiable in both arguments x and t, with bounded derivatives in Q.

Next we consider the existence question for the adjoint problem. Let $\psi \in C_0^\infty(Q)$ be given and let $T>0$ be chosen such that $\psi=0$ for $t>T$. For arbitrary $n\in\mathbb{N}$ consider the linear problem

$$\begin{cases} (\varphi_n)_t + F_n(\varphi_n)_x = \psi & \text{in } Q \\ \varphi_n(x,T) = 0 & \text{for } x \in \mathbb{R} \,. \end{cases}$$

We solve this problem using the method of characteristics. For arbitrary $(x_0,t_0)\in Q$, solve the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = F_n(x_n,t) & \text{for } t \in \mathbb{R}^+ \\ x_n(t_0) = x_0 \,. \end{cases}$$

Because F_n, $\partial F_n/\partial x$ and $\partial F_n/\partial t$ are bounded, there exists a unique solution $x_n(t)=x_n(t;x_0,t_0)$, which satisfies $x_n(t_0;x_0,t_0) = x_0$ and which is continuous differentiable in all its arguments (see standard ODE theory). The solution curve is the characteristic of the equation passing through the point (x_0,t_0). Along this characteristic we solve for φ_n. First we write the equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_n\left(x_n(t;x_0,t_0),t\right) = \psi\left(x_n(t;x_0,t_0),t\right) \,.$$

Then we integrate from T to t_0 and obtain

$$\varphi_n\left(x_n(t_0;x_0,t_0),t_0\right) = \varphi_n(x_0,t_0) = \int_T^{t_0} \psi\left(x_n(t;x_0,t_0),t\right)\mathrm{d}t \,.$$

This function defines a classical solution (C^1) of the partial differential equation which satisfies $\varphi_n(x,t) = 0$ for $t \geqslant T$. Next we show that φ_n has compact support in $\mathbb{R}$, i.e. $\varphi_n \in C_0^1(t \geqslant 0)$.

Because u_n and v_n are uniformly bounded, there exists a positive constant A such that

$$|F_n| \leqslant A \qquad \text{for all } n \in \mathbb{N} \,.$$

Thus the slope of the characteristics is bounded by A. We use this to construct a region $R\subset\overline{Q}$ as in Figure 4.2. Take any point (x_0,t_0) outside R. This means that the corresponding characteristic is outside R. Hence φ_n is constant along the characteristic. Since $\varphi_n(x,T)=0$ this implies that φ_n is zero outside R, showing that $\varphi_n\in C_0^1(t\geqslant 0)$.

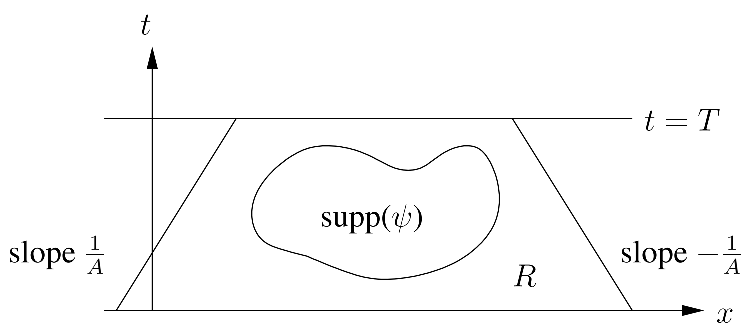


Figure 4.2. *The region R*

Next we discuss the convergence of F_n towards F. We write

$$F - F_n = \int_0^1 \left\{ f'\left(\vartheta u + (1-\vartheta)v\right) - f'\left(\vartheta u_n + (1-\vartheta)v_n\right) \right\} \mathrm{d}\vartheta$$

and use the mean value theorem to obtain

$$F - F_n = \int_0^1 f''(\xi) \big\{ \vartheta(u - u_n) + (1-\vartheta)(v - v_n) \big\} \mathrm{d}\vartheta \ .$$

This gives

$$|F - F_n| \leqslant \frac{1}{2} \sup f''(s) \left\{ |u - u_n| + |v - v_n| \right\} \ .$$

Since $\{u_n\}$ and $\{v_n\}$ converge in $L^1_{\mathrm{loc}}(\overline{Q})$, the same is true for $\{F_n\}$.

It remains to verify the uniform bound on $(\varphi_n)_x$ in $\overline{Q}$. Here the entropy condition will play a crucial role. Let $\alpha > 0$ and $\mu = 2\alpha$. We estimate $(\varphi_n)_x$ in the regions $\mathbb{R} \times [0, \mu]$ and $\mathbb{R} \times [\mu, \infty)$. First consider $t \geqslant \alpha$. The entropy condition implies that

$$u(x,t) - \frac{Ex}{\alpha}$$

is non-increasing in x for $t \geqslant \alpha$. This follows from the observation that for any $x_1 < x_2$ and for all $t \geqslant \alpha$

$$\begin{aligned} 0 &\geqslant \left(u(x_2,t) - \frac{Ex_2}{t} \right) - \left(u(x_1,t) - \frac{Ex_1}{t} \right) \\ &= u(x_2,t) - u(x_1,t) + \frac{E}{t}(x_1 - x_2) \\ &\geqslant u(x_2,t) - u(x_1,t) + \frac{E}{\alpha}(x_1 - x_2) \ . \end{aligned}$$

Set

$$u^* = \begin{cases} u & \text{in } \{t \geqslant \alpha\} \\ 0 & \text{in } \{t < \alpha\} \ . \end{cases}$$

The convolution

$$\rho_n \star \left(u^* - \frac{Ex}{\alpha} \right)$$

is non-increasing in x for all $t \in \mathbb{R}$. Hence for $u_n^* := \rho_n \star u^*$ we have

$$\frac{\partial u_n^*}{\partial x} = \frac{\partial}{\partial x} \left(\rho_n \star u^* \right) \leqslant \frac{\partial}{\partial x} \left(\rho_n \star \frac{Ex}{\alpha} \right) = \rho_n * \frac{E}{\alpha} = \frac{E}{\alpha} \qquad \text{in } \mathbb{R}^2 \ .$$

Similarly, $v_n^* := \rho_n \star v^*$ satisfies

$$\frac{\partial v_n^*}{\partial x} \leqslant \frac{E}{\alpha} \qquad \text{in } \mathbb{R}^2 \ .$$

Choosing n sufficiently large $(n > N(\mu))$ we have $u_n = u_n^*$ and $v_n = v_n^*$ for $t \geqslant \mu$ and thus

$$\frac{\partial u_n}{\partial x} \leqslant \frac{E}{\alpha} \qquad \text{and} \qquad \frac{\partial v_n}{\partial x} \leqslant \frac{E}{\alpha} \qquad \text{in } \mathbb{R} \times [\mu, \infty) \ .$$

Consequently for $n > N(\mu)$

$$\frac{\partial F_n}{\partial x} \leqslant \sup f''(s)\frac{2E}{\mu} =: C(\mu) \qquad \text{in } \mathbb{R}\times[\mu,\infty)\ ,$$

where $C(\mu)$ is independent of n. Note: here we used the convexity of f. Let $(x_0,t_0)\in\mathbb{R}\times[\mu,T]$ and consider the characteristic $x_n(t)=x_n(t;x_0,t_0)$. Set

$$a_n(t) = \frac{\partial x_n}{\partial x_0}\ .$$

Since $x_n(t_0;x_0,t_0)=x_0$ we have $a_n(t_0)=1$. Further

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \frac{\partial}{\partial t}\left(\frac{\partial x_n}{\partial x_0}\right) = \frac{\partial}{\partial x_0}F_n(x_n,t) = \frac{\partial F_n}{\partial x}\frac{\partial x_n}{\partial x_0} = \frac{\partial F_n}{\partial x}a_n\ .$$

Integrating this equation yields

$$a_n(t) = \exp\left\{\int_{t_0}^{t}\frac{\partial F_n}{\partial x}\left(x_n(\tau),\tau\right)\mathrm{d}\tau\right\} \qquad \text{for } n\in\mathbb{N} \text{ and } t\in\mathbb{R}$$

and thus for $\mu=t_0<t<T$ and $n>N(\mu)$

$$0 < a_n(t) < \exp\left\{C(\mu)(T-\mu)\right\}\ .$$

Since

$$\frac{\partial\varphi_n}{\partial x} = \int_T^t \frac{\partial\psi}{\partial x}\left(x_n(\tau),\tau\right)\frac{\partial x_n}{\partial x}\mathrm{d}\tau = \int_T^t\frac{\partial\psi}{\partial x}\left(x_n(\tau),\tau\right)a_n(\tau)\mathrm{d}\tau$$

we obtain the bound

$$\left|\frac{\partial\varphi_n}{\partial x}\right| \leqslant K(\mu) \qquad \text{in } \mathbb{R}\times[\mu,T] \text{ and for } n>N(\mu)\ .$$

Next we consider the strip $\mathbb{R}\times[0,\mu]$. For fixed $t\in[0,T]$, the support of $\varphi_n(\cdot,t)$ is independent of n. Hence the variation

$$V_t(\varphi_n) := \int_{\mathbb{R}}\left|\frac{\partial\varphi_n}{\partial x}\right|\mathrm{d}x$$

is bounded by

$$V_t(\varphi) \leqslant \tilde{K}(\mu) \qquad \text{for } t\in[\mu,T] \text{ and for } n>N(\mu)\ .$$

Now choose μ as in Figure 4.3: i.e. $\psi=0$ in $\mathbb{R}\times[0,\mu]$. For this choice, φ_n is constant along the characteristics if $0\leqslant t\leqslant\mu$. Since characteristics cannot intersect (F_n is locally Lipschitz) this implies

$$V_t(\varphi_n) = V_\mu(\varphi_n) \leqslant \tilde{K}(\mu) \qquad \text{for } 0\leqslant t\leqslant\mu \text{ and } n>N(\mu)\ .$$

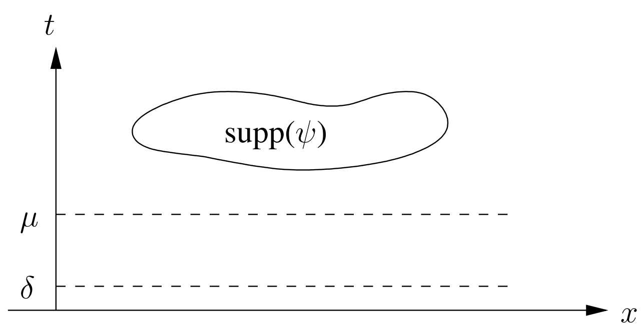


Figure 4.3. *Support of* ψ

We now are in the position to complete the proof. For $n > N(\mu)$ we write

$$\begin{aligned}\left|\int_Q (u-v)\psi\right| &\leqslant \int_Q |u-v|\,|F_n-F|\,|(\varphi_n)_x| \\ &= \int_{\{0<t<\delta\}} |u-v|\,|F_n-F|\,|(\varphi_n)_x| + \int_{\{t\geqslant\delta\}} |u-v|\,|F_n-F|\,|(\varphi_n)_x| \\ &=: I_1 + I_2\,.\end{aligned}$$

First choose $\varepsilon > 0$. Then choose δ sufficiently small ($\delta < \mu$), so that

$$I_1 < 4MA \int_{\{0<t<\delta\}} |(\varphi_n)_x| = 4MA \int_0^\delta V_t(\varphi_n)\mathrm{d}t \leqslant 4MA\tilde{K}(\mu)\delta < \frac{\varepsilon}{2}\,.$$

The second term we estimate using the choice of δ:

$$I_2 \leqslant 2M \int_{\{t\geqslant\delta\}} |F_n-F|\,|(\varphi_n)_x| \leqslant 2MK(\delta) \int_{\{t\geqslant\delta\}\cap R} |F_n-F| < \frac{\varepsilon}{2}$$

for n sufficiently large. Hence

$$\left|\int_Q (u-v)\,\psi\,\right| < \varepsilon \qquad \text{for all } \varepsilon > 0$$

and thus

$$\int_Q (u-v)\psi = 0\,,$$

implying the uniqueness. □

Remark 4.2. *In the last step of the proof we first choose μ as in Figure 4.3. Then for any $\varepsilon > 0$, take $n > N(\mu)$ and $\delta < \mu$ so that $|I_1| < \varepsilon/2$. For this fixed δ, repeat the estimates with $\alpha = 2\delta$ and $\mu = \delta$ to obtain $\left|\frac{\partial\varphi_n}{\partial x}\right| < K(\delta)$ in $\mathbb{R}\times[\delta,T]$ for $n > N(\delta)$. Finally, choose n sufficiently large so that $|I_2| < \varepsilon/2$.*

5 Existence of the entropy solution

In this section we prove the following theorem.

Theorem 5.1. *Suppose* $u_0 \in L^\infty(\mathbb{R})$, $f \in C^2(\mathbb{R})$ *and* $f''(s) > 0$ *for* $|s| \leqslant \|u_0\|_{L^\infty(\mathbb{R})} =: M$. *Then there exists a weak solution* u *of* (P) *with the following properties:*

(*i*) $\|u\|_{L^\infty(Q)} \leqslant \|u_0\|_{L^\infty(\mathbb{R})}$;

(*ii*) *Entropy condition: there exists a constant* $E > 0$, *depending on* M, $\mu := \min\limits_{|s|\leqslant M} f''(s)$ *and* $A := \max\limits_{|s|\leqslant M} |f'(s)|$ *such that for any* $a > 0$ *and* $(x,t) \in Q$

$$\frac{u(x+a,t) - u(x,t)}{a} \leqslant \frac{E}{t}\ ; \tag{5.1}$$

(*iii*) *If* $v_0 \in L^\infty(\mathbb{R})$ *with* $\|v_0\|_{L^\infty(\mathbb{R})} \leqslant \|u_0\|_{L^\infty(\mathbb{R})}$ *and* $v \in L^\infty(Q)$ *is the correspondingly constructed solution of* (P) *with initial data* v_0, *then* $u_0 \leqslant v_0$ *on* $\mathbb{R}$ *implies* $u \leqslant v$ *in* Q;

(*iv*) *Continuous dependence. Let* u *and* v *be as above. Then for every pair* $x_1, x_2 \in \mathbb{R}$ *and* $t > 0$ *we have*

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)| \mathrm{d}x \leqslant \int_{x_1 - At}^{x_2 + At} |u_0(x) - v_0(x)| \mathrm{d}x\ . \tag{5.2}$$

Before we start the proof, we point out some consequences of property (*iv*):

- Uniqueness for correspondingly constructed solutions;
- If u_0 has bounded support in $\mathbb{R}$, then $u(\cdot,t)$ has bounded support in $\mathbb{R}$ for all $t > 0$. To prove this we take $v_0 = v = 0$ in (5.2) and consider a situation as in Figure 5.1. By construction we find that
$$u(x,t) = 0 \text{ for } t > 0,\ x > a_2 + At \text{ and for } t > 0,\ x < a_1 - At\ .$$

We prove the theorem by the method of finite differences. In this method we discretize the differential equation on a grid in the domain Q. Let

$$h = \Delta t \qquad \text{and} \qquad l = \Delta x$$

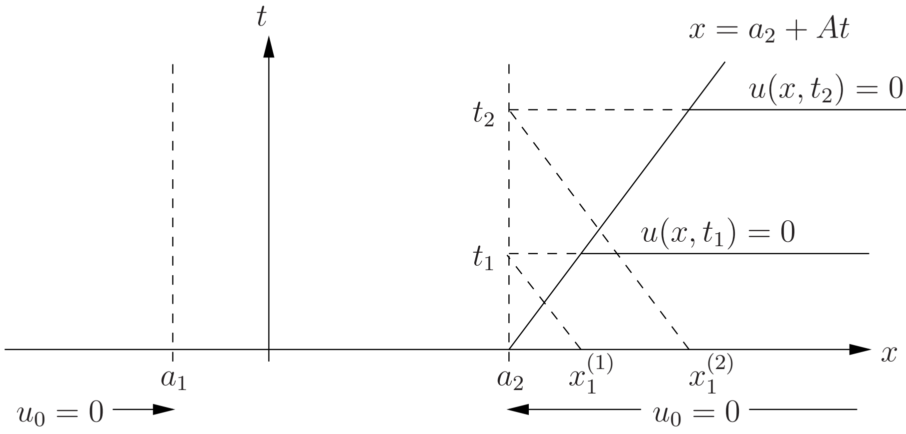


Figure 5.1. *Bounded support*

(grid parameters). Then define a grid according to

$$t = kh \qquad \text{and} \qquad x = nl ,$$

where $k \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$ and consider the difference approximation

$$u_n^{k+1} = \frac{1}{2}\left(u_{n+1}^k + u_{n-1}^k\right) - \frac{h}{2l}\left\{f\left(u_{n+1}^k\right) - f\left(u_{n-1}^k\right)\right\} \tag{5.3}$$

for $k \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, with the initial condition

$$u_n^0 := u_0(nl) \qquad \left(\text{or } \frac{1}{l}\int_{(n-\frac{1}{2})l}^{(n+\frac{1}{2})l} u_0(s)\mathrm{d}s\right) \qquad \text{for } n \in \mathbb{Z} . \tag{5.4}$$

The finite difference scheme (5.3) is called the *Lax-scheme*. Observe that u_n^k for $n-k$ even is computed independently of u_n^k for $n-k$ odd. The Lax-scheme is not very accurate. For computational purposes it contains too much numerical dispersion. This one sees using a formal argument with Taylor expansions. With the notation $u_i^j = u(il, jh)$ one finds:

$$\begin{aligned}
u_n^{k+1} &= u_n^k + h\frac{\partial u}{\partial t} + \mathcal{O}(h^2)\\
u_{n+1}^k &= u_n^k + l\frac{\partial u}{\partial x} + \frac{l^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{l^3}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}(l^4)\\
u_{n-1}^k &= u_n^k - l\frac{\partial u}{\partial x} + \frac{l^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{l^3}{3!}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}(l^4)\\
f_{n+1}^k &:= f(u_{n+1}^k) = f_n^k + l\frac{\partial f}{\partial x} + \frac{l^2}{2}\frac{\partial^2 f}{\partial x^2} + \mathcal{O}(l^3)\\
f_{n-1}^k &:= f(u_{n-1}^k) = f_n^k - l\frac{\partial f}{\partial x} + \frac{l^2}{2}\frac{\partial^2 f}{\partial x^2} + \mathcal{O}(l^3) .
\end{aligned}$$

Substituting these expressions into (5.3) gives

$$h\frac{\partial u}{\partial t} + \mathcal{O}(h^2) = \frac{l^2}{2}\frac{\partial^2 u}{\partial x^2} + \mathcal{O}(l^4) - \frac{h}{2l}\left\{2l\frac{\partial f}{\partial x} + \mathcal{O}(l^3)\right\}$$

or

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \frac{l^2}{2h}\frac{\partial^2 u}{\partial x^2} + \mathcal{O}(l^2) + \mathcal{O}(h) \;.$$

The additional diffusion term is induced by the discretization. It is called *numerical dispersion*. For theoretical purposes, however, it is well-suited because it has certain monotonicity and stability properties, and because it satisfies the entropy condition (5.1). Other difference schemes, having similar properties, are discussed by CRANDELL & MAJDA [17].

Below we first explain the properties of the Lax-scheme. Later we shall define approximating functions and pass to the limit for $l, h \to 0$. Throughout this section we choose the grid parameter l and h such that

$$Ah/l \leqslant 1 \qquad \textit{(Stability condition)} \;. \tag{5.5}$$

Proposition 5.2. *(Comparison principle). Suppose u_n^k and v_n^k are solutions of* (5.3), (5.4) *with initial conditions u_n^0 and v_n^0, respectively, and*

$$-M \leqslant u_n^0 \leqslant v_n^0 \leqslant M \qquad \text{for } n \in \mathbb{Z} \;.$$

Then

$$-M \leqslant u_n^k \leqslant v_n^k \leqslant M \qquad \text{for } n \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+ \;.$$

Proof. We prove that for any $k \in \mathbb{Z}^+$

$$-M \leqslant u_n^k \leqslant v_n^k \leqslant M \qquad \Rightarrow \qquad -M \leqslant u_n^{k+1} \leqslant v_n^{k+1} \leqslant M \qquad \text{for all } n \in \mathbb{Z} \;.$$

Subtracting the difference equations for u_n^k and v_n^k gives

$$\begin{aligned} u_n^{k+1} - v_n^{k+1} \;&=\; \frac{u_{n+1}^k - v_{n+1}^k}{2} - \frac{h}{2l}\left\{f(u_{n+1}^k) - f(v_{n+1}^k)\right\} \\ &\quad + \frac{u_{n-1}^k - v_{n-1}^k}{2} + \frac{h}{2l}\left\{f(u_{n-1}^k) - f(v_{n-1}^k)\right\} \\ &=\; \frac{u_{n+1}^k - v_{n+1}^k}{2}\left\{1 - \frac{h}{l} f'(\theta_{n+1}^k)\right\} \\ &\quad + \frac{u_{n-1}^k - v_{n-1}^k}{2}\left\{1 + \frac{h}{l} f'(\theta_{n-1}^k)\right\} \;, \end{aligned} \tag{5.6}$$

where $u_{n+1}^k \leqslant \theta_{n+1}^k \leqslant v_{n+1}^k$ and $u_{n-1}^k \leqslant \theta_{n-1}^k \leqslant v_{n-1}^k$. By assumption we have $-M \leqslant \theta_{n+1}^k,\ \theta_{n-1}^k \leqslant M$ and thus

$$|f'(\theta_{n+1}^k)|,\ |f'(\theta_{n-1}^k)| \leqslant A \;.$$

We use (5.5) to conclude

$$u_n^{k+1} \leqslant v_n^{k+1} \;.$$

A comparison with the constant solutions $\pm M$ gives the desired statement. The proof is completed by induction. □

Proposition 5.3. *Suppose* $-M \leqslant u_n^0,\ v_n^0 \leqslant M$ *for* $n \in \mathbb{Z}$. *If* $u^0 - v^0 \in \ell^1(\mathbb{Z})$*, then*

$$u^k - v^k \in \ell^1(\mathbb{Z})$$

and

$$\sum_{n=-\infty}^{\infty} (u_n^k - v_n^k) = \sum_{n=-\infty}^{\infty} (u_n^0 - v_n^0) \qquad \text{for all } k \in \mathbb{Z}^+ .$$

Proof. The first assertion is a direct consequence of expression (5.6). To prove the second one we take an arbitrary $N \in \mathbb{N}$ and $k \in \mathbb{Z}^+$ and compute

$$\sum_{n=-N}^{N} (u_n^{k+1} - v_n^{k+1}) \;=$$

$$\frac{1}{2}\Big\{(u_{-N-1}^k - v_{-N-1}^k) + (u_{-N}^k - v_{-N}^k) + (u_N^k - v_N^k) + (u_{N+1}^k - v_{N+1}^k)\Big\} +$$

$$\sum_{n=-N+1}^{N-1} (u_n^k - v_n^k) +$$

$$\frac{h}{2l}\Big\{f(u_{-N-1}^k) - f(v_{-N-1}^k) + f(u_{-N}^k) - f(v_{-N}^k) - f(u_N^k) + f(v_N^k) - f(u_{N+1}^k) + f(v_{N+1}^k)\Big\} .$$

Letting $N \to \infty$ gives

$$\sum_{n=-\infty}^{\infty} (u_n^{k+1} - v_n^{k+1}) = \sum_{n=-\infty}^{\infty} (u_n^k - v_n^k) , \tag{5.7}$$

from which the result immediately follows. □

The following stability result is an application of a lemma of CRANDELL–TARTAR [18].

Proposition 5.4. *(Stability 1) Suppose* $-M \leqslant u_n^0, v_n^0 \leqslant M$ *for* $n \in \mathbb{Z}$. *If* $u^0 - v^0 \in \ell^1(\mathbb{Z})$*, then*

$$\|u^k - v^k\|_{\ell^1(\mathbb{Z})} \leqslant \|u^0 - v^0\|_{\ell^1(\mathbb{Z})} \qquad \textit{for all } k \in \mathbb{Z}^+ .$$

Proof. We introduce the notation

$$u^{k+1} = T(u^k) \qquad \text{for } k \in \mathbb{Z}^+ ,$$

where $T : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is defined by the right hand side of (5.3). From Proposition 5.2 it follows that T is a monotone operator, i.e.

$$u^k \geqslant v^k \qquad \Rightarrow \qquad T(u^k) \geqslant T(v^k) .$$

For $w^k := \max\{u^k, v^k\}$ this implies

$$T(w^k) - T(u^k) \geqslant 0$$

and

$$T(w^k) - T(v^k) \geqslant 0 .$$

Hence

$$T(u^k) - T(v^k) \leqslant T(w^k) - T(v^k) \ .$$

Because the right hand side of this inequality is nonnegative we also have

$$(T(u^k) - T(v^k))_+ \leqslant T(w^k) - T(v^k) \ , \tag{5.8}$$

where $(\cdot)_+$ denotes the positive part: i.e. $(s)_+ = \max\{0, s\}$. Now for any $a, b \in \mathbb{R}$ we have

$$\max\{a, b\} - b = (a-b)_+ \leqslant |a-b| \ .$$

Hence

$$0 \leqslant w^k - v^k = (u^k - v^k)_+ \leqslant |u^k - v^k| \ . \tag{5.9}$$

Thus if $u^k - v^k \in \ell^1(\mathbb{Z})$ then also $w^k - v^k \in \ell^1(\mathbb{Z})$ and equality (5.7), applied to w^k and v^k gives

$$\sum_{n=-\infty}^{\infty} (T(w_n^k) - T(v_n^k)) = \sum_{n=-\infty}^{\infty} (w_n^k - v_n^k) = \sum_{n=-\infty}^{\infty} (u_n^k - v_n^k)_+ \ ,$$

where we used (5.9). Thus the summation of (5.8) results in

$$\sum_{n=-\infty}^{\infty} (T(u_n^k) - T(v_n^k))_+ \leqslant \sum_{n=-\infty}^{\infty} (u_n^k - v_n^k)_+ \ . \tag{5.10}$$

Similary one finds

$$\sum_{n=-\infty}^{\infty} (T(v_n^k) - T(u_n^k))_+ \leqslant \sum_{n=-\infty}^{\infty} (v_n^k - u_n^k)_+ \ . \tag{5.11}$$

Because

$$|a| = (a)_+ + (-a)_+ \qquad \text{for } a \in \mathbb{R} \ ,$$

it follows from (5.10) and (5.11) that

$$\|T(v^k) - T(u^k)\|_{\ell^1(\mathbb{Z})} \leqslant \|v^k - u^k\|_{\ell^1(\mathbb{Z})} \ .$$

The proof is completed by induction. □

Next we give a second stability result, which is the discrete version of inequality (5.2).

Proposition 5.5. *(Stability 2) Suppose $-M \leqslant u_n^0, v_n^0 \leqslant M$ for $n \in \mathbb{Z}$. Then for any $N \in \mathbb{N}$*

$$\sum_{|n| \leqslant N} |u_n^k - v_n^k| \leqslant \sum_{|n| \leqslant N+k} |u_n^0 - v_n^0|$$

holds for all $k \in \mathbb{Z}^+$.

Proof. For $k \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$ set $w_n^k := u_n^k - v_n^k$. Again we use expression (5.6) to obtain

$$w_n^{k+1} = \left\{\frac{1}{2} - \frac{h}{2l} f'(\theta_{n+1}^k)\right\} w_{n+1}^k + \left\{\frac{1}{2} + \frac{h}{2l} f'(\theta_{n-1}^k)\right\} w_{n-1}^k ,$$

where the coefficients of w_{n+1}^k and w_{n-1}^k are positive. Hence for any $N \in \mathbb{N}$ and $k \in \mathbb{Z}^+$ we have

$$\begin{aligned}
\sum_{|n|\leqslant N} \left|w_n^{k+1}\right| &\leqslant \sum_{|n|\leqslant N} \left\{\frac{1}{2} - \frac{h}{2l} f'(\theta_{n+1}^k)\right\} \left|w_{n+1}^k\right| + \sum_{|n|\leqslant N} \left\{\frac{1}{2} + \frac{h}{2l} f'(\theta_{n-1}^k)\right\} \left|w_{n-1}^k\right| \\
&= \sum_{m=-N+1}^{N+1} \left\{\frac{1}{2} - \frac{h}{2l} f'(\theta_m^k)\right\} \left|w_m^k\right| + \sum_{m=-N-1}^{N-1} \left\{\frac{1}{2} + \frac{h}{2l} f'(\theta_m^k)\right\} \left|w_m^k\right| \\
&\leqslant \sum_{|m|\leqslant N+1} \left\{\frac{1}{2} - \frac{h}{2l} f'(\theta_m^k)\right\} \left|w_m^k\right| + \sum_{|m|\leqslant N+1} \left\{\frac{1}{2} + \frac{h}{2l} f'(\theta_m^k)\right\} \left|w_m^k\right| \\
&= \sum_{|n|\leqslant N+1} |w_n^k| \, .
\end{aligned} \tag{5.12}$$

We proceed by induction. Inequality (5.12) shows that the statement of the proposition is true for $k=1$. Now suppose for arbitrary $k \in \mathbb{N}$ that

$$\sum_{|n|\leqslant N} \left|w_n^k\right| \leqslant \sum_{|n|\leqslant N+k} |w_n^0| \, .$$

Then with (5.12) we obtain

$$\sum_{|n|\leqslant N} \left|w_n^{k+1}\right| \leqslant \sum_{|n|\leqslant N+1} \left|w_n^k\right| \leqslant \ldots \leqslant \sum_{|n|\leqslant N+k+1} |w_n^0| \, ,$$

which proves the result. ☐

In the following propositions we assume that $u_0 \in \ell^\infty(\mathbb{Z})$ such that $-M \leqslant u_n^0 \leqslant M$ for all $n \in \mathbb{Z}$ and that u_n^k is the solution of the difference equation (5.3).

Proposition 5.6. *(Entropy condition) Let* $c = \min\left\{\frac{\mu}{2}, \frac{A}{4M}\right\}$. *Then for all* $n \in \mathbb{Z}$ *and* $k \in \mathbb{N}$

$$\frac{u_n^k - u_{n-2}^k}{2l} \leqslant \frac{E}{kh} \qquad \textit{with } E = \frac{1}{c} \, .$$

Proof. Let $z_n^k := (u_n^k - u_{n-2}^k)/2l$ for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. Using the difference equation we find

$$z_n^{k+1} = \frac{1}{2}\big(z_{n+1}^k + z_{n-1}^k\big) - \frac{h}{4l^2}\Big\{\Big(f(u_{n+1}^k) - f(u_{n-1}^k)\Big) - \Big(f(u_{n-1}^k) - f(u_{n-3}^k)\Big)\Big\} \, .$$

Next we expand

$$f(u_{n+1}^k) = f(u_{n-1}^k) + f'(u_{n-1}^k) 2l z_{n+1}^k + f''(\theta_n^k) 2l^2 (z_{n+1}^k)^2$$

and

$$f(u_{n-3}^k) = f(u_{n-1}^k) - f'(u_{n-1}^k) 2l z_{n-1}^k + f''(\theta_{n-2}^k) 2l^2 (z_{n-1}^k)^2 \, .$$

Substitution gives

$$z_n^{k+1} = \left\{\frac{1}{2} - \frac{h}{2l} f'(u_{n-1}^k)\right\} z_{n+1}^k + \left\{\frac{1}{2} + \frac{h}{2l} f'(u_{n-1}^k)\right\} z_{n-1}^k$$
$$- \frac{h}{2}\left\{ f''(\theta_n^k)(z_{n+1}^k)^2 + f''(\theta_{n-2}^k)(z_{n-1}^k)^2 \right\} ,$$

which implies

$$z_n^{k+1} \leqslant \left\{\frac{1}{2} - \frac{h}{2l} f'(u_{n-1}^k)\right\} z_{n+1}^k + \left\{\frac{1}{2} + \frac{h}{2l} f'(u_{n-1}^k)\right\} z_{n-1}^k$$
$$- hc\left\{ (z_{n+1}^k)^2 + (z_{n-1}^k)^2 \right\} . \qquad (5.13)$$

Now introduce for $k \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$

$$\tilde{z}_n^k = \max\left\{ z_{n+1}^k, z_{n-1}^k, 0 \right\} \geqslant 0 .$$

Using this in (5.13) gives

$$z_n^{k+1} \leqslant \left\{\frac{1}{2} - \frac{h}{2l} f'(u_{n-1}^k)\right\} \tilde{z}_n^k + \left\{\frac{1}{2} + \frac{h}{2l} f'(u_{n-1}^k)\right\} \tilde{z}_n^k - hc(\tilde{z}_n^k)^2$$

or

$$z_n^{k+1} \leqslant \tilde{z}_n^k - hc(\tilde{z}_n^k)^2 . \qquad (5.14)$$

Next we estimate

$$z_n^k \leqslant \left| z_n^k \right| = \left| \frac{u_n^k - u_{n-2}^k}{2l} \right| \leqslant \frac{M}{l} \leqslant \frac{M}{Ah} \leqslant \frac{M}{4Mch} = \frac{1}{4ch} ,$$

where we used $\dfrac{hA}{l} \leqslant 1$ and $\dfrac{A}{4M} \geqslant c$. Then, introducing

$$M^k := \sup_{n \in \mathbb{Z}} \left\{ \tilde{z}_n^k \right\} \qquad (k \in \mathbb{Z}^+) ,$$

we obtain

$$\tilde{z}_n^k \leqslant M^k \leqslant \frac{1}{4ch} < \frac{1}{2ch} . \qquad (5.15)$$

Because the function

$$\phi(y) = y - chy^2 , \qquad y \geqslant 0$$

is strictly increasing for $0 < y < 1/(2ch)$, we find from (5.14) and (5.15)

$$z_n^{k+1} \leqslant \phi(\tilde{z}_n^k) \leqslant \phi(M^k) \qquad \text{for all } n \in \mathbb{Z} .$$

Consequently

$$M^{k+1} \leqslant \phi(M^k)$$

and

$$\frac{M^{k+1} - M^k}{h} \leqslant -c(M^k)^2 \qquad \text{for all } k \in \mathbb{Z}^+ \ .$$

From this inequality we see

$$\begin{aligned} M^0 = 0 \qquad &\Rightarrow \qquad M^k = 0 \text{ for all } k \in \mathbb{N} \\ &\Rightarrow \qquad z_n^k \leqslant 0 \text{ for all } n \in \mathbb{Z} \text{ and } k \in \mathbb{N} \ . \end{aligned}$$

Hence if $M^0 = 0$ (this occurs for example if u_0 is non-increasing), the proposition is true. If $M^0 > 0$, we compare the solution M^k of the difference inequality with the solution of the differential equation

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = -cw^2 \quad \text{for } t > 0 \ , \\ w(0) = M^0 \ . \end{cases}$$

The solution is given by

$$w(t) = \frac{1}{ct + \frac{1}{M^0}} \qquad \text{for } t \geqslant 0 \ .$$

We prove below that

$$M^k \leqslant w(kh) \qquad \text{for all } k \in \mathbb{Z}^+ \ . \tag{$*$}$$

Inequality $(*)$ implies

$$M^k \leqslant \frac{1}{ckh + \frac{1}{M^0}} < \frac{1}{ckh} = \frac{E}{kh} \ ,$$

which gives

$$\frac{u_n^k - u_{n-2}^k}{2l} = z_n^k \leqslant \tilde{z}_n^k \leqslant M^k < \frac{E}{hk} \ ,$$

and concludes the proof of the proposition. □

Proof of inequality $(*)$. We show that

$$M^k \leqslant w(kh) \Rightarrow M^{k+1} \leqslant w((k+1)h) \ .$$

From (5.15) it follows that $M^0 \leqslant \frac{1}{4ch}$ and thus

$$w(kh) = \frac{1}{ckh + \frac{1}{M_0}} \leqslant \frac{1}{ckh + 4ch} < \frac{1}{2ch} \ .$$

Since ϕ is strictly increasing on $[0, \frac{1}{2ch}]$ we obtain

$$\phi(M^k) \leqslant \phi\left(w(kh)\right) \ ,$$

which implies

$$M^{k+1} \leqslant \phi\left(w(kh)\right) = w(kh) - ch\left(w(kh)\right)^2 = w(kh) + hw'(kh) \leqslant w\left((k+1)h\right) \ ,$$

where we used the convexity of the function w. □

As in the continuous case, the entropy inequality implies a bound, local in time, on the variation in space. Before we give the proposition, we write

$$S^1 \cup S^2 = \left\{(n,k) | n \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^+\right\} \ ,$$

with

$$S^1 = \{(n,k) | n-k \text{ even}\}$$

and

$$S^2 = \{(n,k) | n-k \text{ odd}\} \ .$$

Since the u_n^k with $(n,k) \in S^1$ are computed independently from the u_n^k with $(n,k) \in S^2$, we expect a result either for $(n,k) \in S^1$ or for $(n,k) \in S^2$. This corresponds also to the entropy condition which relates u_n^k and u_{n-2}^k.

Proposition 5.7. *(Space estimate) Let $\alpha, L > 0$. Choose $k \in \mathbb{N}$ such that $kh > \alpha$. Then there exists a constant $C = C(L, \alpha, M)$ such that*

$$\sum_{|n| \leqslant \frac{L}{l}} |u_{n+2}^k - u_n^k| \leqslant C \ .$$

Proof. Let $k \in \mathbb{N}$ with $kh > \alpha$ and let

$$v_n^k := u_n^k - c_1 n l \qquad \text{for all } n \in \mathbb{Z}$$

with $c_1 > \dfrac{E}{\alpha}$. Then

$$v_{n+2}^k - v_n^k < 0 \qquad \text{for all } n \in \mathbb{Z} \ .$$

Hence

$$\begin{aligned}
\sum_{|n| \leqslant \frac{L}{l}} \left| u_{n+2}^k - u_n^k \right| &\leqslant \sum_{|n| \leqslant \frac{L}{l}} \left| v_{n+2}^k - v_n^k \right| + \sum_{|n| \leqslant \frac{L}{l}} 2c_1 l \\
&= \sum_{|n| \leqslant \frac{L}{l}} \left(v_n^k - v_{n+2}^k \right) + 2 \sum_{|n| \leqslant \frac{L}{l}} c_1 l = \sum_{|n| \leqslant \frac{L}{l}} \left(u_n^k - u_{n+2}^k \right) + 4 \sum_{|n| \leqslant \frac{L}{l}} c_1 l \\
&\leqslant 4M + 10 c_1 L =: C \ ,
\end{aligned}$$

where we used $L \geqslant 2l$. □

Next we present a time estimate. With $u^k \in \ell^1_{\mathrm{loc}}(\mathbb{Z})$, we show that the corresponding norm is locally Lipschitz continuous in k.

Definition 5.8. *Let $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be such that $f(\cdot, t) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for all $t > 0$. Then f is called locally L^1_{loc}-Lipschitz continuous if for every compact set $B \subset \mathbb{R}$ and for every positive number $t^* > 0$, there exists a constant $K = K(B, t^*)$ such that*

$$\|f(\cdot, t_1) - f(\cdot, t_2)\|_{L^1(B)} \leqslant K |t_1 - t_2|$$

for all $t_1, t_2 \geqslant t^*$.

Proposition 5.9. *(Time estimate) Choose $\alpha, L > 0$ and let $\delta := h/l$. Further let $k, p \in \mathbb{N}$ such that $k > p$ and $ph > \alpha$. Then there exists a constant $C^* = C^*(\alpha, L, M, \delta)$ such that for $k-p$ even*

$$\sum_{|n| \leqslant L/l} \left|u_n^k - u_n^p\right| l \leqslant C^* (k-p) h .$$

Similarly we find

$$\sum_{|n| \leqslant L/l} \left|u_n^k - u_{n+1}^p\right| l \leqslant C^* (k-p) h$$

for $k-p$ odd.

Proof. We prove here only the case $k-p$ even. We apply a Taylor expansion in the finite difference equation (5.3). This gives

$$u_n^{k+1} = \left\{\frac{1}{2} + f'\left(\theta_n^k\right)\frac{h}{2l}\right\} u_{n-1}^k + \left\{\frac{1}{2} - f'\left(\theta_n^k\right)\frac{h}{2l}\right\} u_{n+1}^k ,$$

which we write as

$$u_n^{k+1} = a u_{n-1}^k + b u_{n+1}^k \qquad (a+b=1,\ a,b \geqslant 0) .$$

This we repeat and obtain

$$u_n^{k+1} = A u_{n-2}^{k-1} + B u_n^{k-1} + C u_{n+2}^{k-1} \qquad (A+B+C=1,\ A,B,C \geqslant 0) .$$

Hence

$$u_n^{k+1} - u_n^{k-1} = A\left(u_{n-2}^{k-1} - u_n^{k-1}\right) + C\left(u_{n+2}^{k-1} - u_n^{k-1}\right)$$

or

$$\left|u_n^{k+1} - u_n^{k-1}\right| \leqslant \left|u_{n-2}^{k-1} - u_n^{k-1}\right| + \left|u_{n+2}^{k-1} - u_n^{k-1}\right| .$$

We apply Proposition 5.7 and obtain

$$\sum_{|n| \leqslant L/l} \left|u_n^{k+1} - u_n^{k-1}\right| l \ \leqslant\ Cl \qquad \text{as long as } (k-1)h > \alpha .$$

Because $k-p$ is even we have

$$u_n^k - u_n^p = \sum_{\substack{i=p \\ (i-p \text{ even})}}^{k-2} \left(u_n^{i+2} - u_n^i\right) .$$

Hence

$$\left|u_n^k - u_n^p\right| \ \leqslant \sum_{\substack{i=p \\ (i-p \text{ even})}}^{k-2} \left|u_n^{i+2} - u_n^i\right| ,$$

and consequently

$$\sum_{|n| \leqslant L/l} \left|u_n^k - u_n^p\right| l \ \leqslant \sum_{\substack{i=p \\ (i-p \text{ even})}}^{k-2} \left\{\sum_{|n| \leqslant \frac{L}{l}} \left|u_n^{i+2} - u_n^i\right| l\right\} \leqslant \sum_{\substack{i=p \\ (i-p \text{ even})}}^{k-2} Cl = \frac{k-p}{2} Cl = \frac{C}{2\delta}(k-p)h ,$$

which gives the desired inequality with $C^* := C/2\delta$. ☐

Remark 5.10. *The results of Propositions 5.7 and 5.9 also hold if we sum over points n such that $(n,k) \in S^i$ (i.e. $n-k$ even $(i=1)$ and $n-k$ odd $(i=2)$): one has with similar constants*

$$\sum_{|n|\leqslant L/l,\ (n,k)\in S^i} \left|u_{n+2}^k - u_n^k\right| \leqslant C\ , \tag{5.16}$$

and

$$\sum_{|n|\leqslant L/l,\ (n,k)\in S^1} \left|u_n^k - u_n^p\right| 2l \leqslant C^*\,(k-p)\,h \tag{5.17a}$$

for $k-p$ even, or

$$\sum_{|n|\leqslant L/l,\ (n,k)\in S^2} \left|u_n^k - u_{n+1}^p\right| 2l \leqslant C^*\,(k-p)\,h \tag{5.17b}$$

for $k-p$ odd.

Next we prove convergence towards a weak solution of (P). Let $h, l > 0$ be such that

$$\frac{h}{l} = \delta$$

is fixed and

$$\frac{Ah}{l} \leqslant 1 \qquad \text{(stability condition).}$$

We consider the family of functions $\{U_{h,l}\}_{h,l>0}$ with $U_{h,l} : \overline{Q} \to \mathbb{R}$ defined by

$$U_{h,l}(x,t) = u_n^k \qquad \text{for } nl \leqslant x < (n+2)l \text{ and } kh \leqslant t < (k+1)h\ ,$$

where $(n,k) \in S^1$ (see Figure 5.2). From the comparison principle (Proposition 5.2) we obtain

$$\|U_{h,l}\|_{L^\infty(Q)} \leqslant M \qquad \text{uniformly in } h, l > 0\ .$$

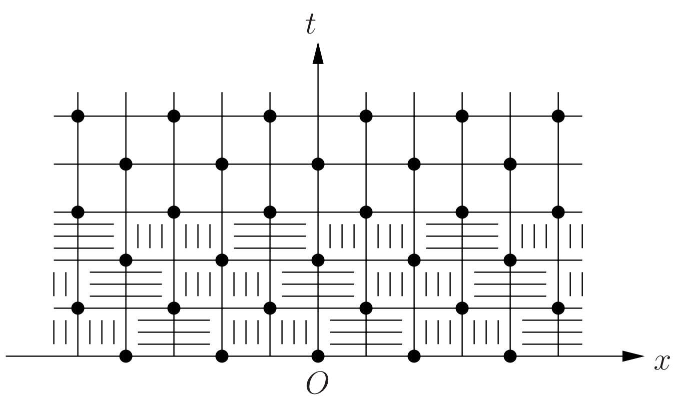


Figure 5.2. *Construction of $U_{n,l}$*

Now fix $t_0 > 0$. Then for each $n \in \mathbb{N}$ we have for the approximation and its x-variation the estimates

$$\|U_{h,l}(\cdot, t_0)\|_{L^\infty([-n,n])} \leqslant M$$

and

$$\mathrm{V}_{-n,n} U_{h,l}(\cdot, t_0) \leqslant C(n, t_0, M) \ ,$$

which hold uniformly in $l > 0$ and $0 < h < t_0$. The second inequality is a direct consequence of Proposition 5.7, see also Remark 5.10.

In order to pass to the limit for $h, l \to 0$ we need the following result.

Lemma 5.11. *(Helly's theorem) Suppose there exist positive constants M, C and a family of functions $F = \{f : [a, b] \to \mathbb{R}\}$, with $[a, b] \subset \mathbb{R}$ fixed, such that*

$$(i) \quad \|f\|_{L^\infty(a,b)} \leqslant M,$$
$$(ii) \quad V_{a,b}\, f \leqslant C,$$

for all $f \in F$. Then given any sequence $\{f_n\} \subset F$, there exists a subsequence $\{f_{n_i}\}$ and a function $\varphi \in F$ such that for each $x \in [a, b]$

$$f_{n_i}(x) \to \varphi(x) \qquad \text{as } n_i \to \infty \ ;$$

i.e. pointwise convergence along a subsequence.

Proof. See TAYLOR [70]. □

Since $l = h/\delta$, it is convenient to introduce the notation

$$U_h(x, t) := U_{h,h/\delta}(x, t) \ .$$

We apply Lemma 5.11 to the family $\{U_h(\cdot, t_0)\}_{0<h<t_0}$. For instance, taking $f_n := U_{\frac{t_0}{n+1}}(\cdot, t_0)$, we obtain a subsequence which converges for each $x \in [-n, n]$. Let us denote this subsequence by $\{U_{h_i}(\cdot, t_0)\}$, where $h_i \to 0$ as $i \to \infty$.

Next we consider subsequences of the subsequence to extend the domain of convergence. First we apply a standard diagonal process to extract a subsequence which converges pointwise on $\mathbb{R}$. Next we fix $T > 0$ and select a countable dense subset $E \subset (0, T)$. Again we apply the diagonal process to obtain a subsequence which converge pointwise on $\mathbb{R} \times E$. As a result we have constructed a sequence, denoted again by $\{U_{h_i}\}$, such that $U_{h_i}(x, t)$ converges for each $(x, t) \in \mathbb{R} \times E$ as $i \to \infty$ (with $h_i \to 0$).

We first show

Proposition 5.12. *Let $X > 0$. Then for any $t \in (0, T]$ the sequence $\{U_i := U_{h_i}\}_{i=1}^\infty$ is a Cauchy sequence in $L^1([-X, X])$.*

Proof. Let $t \in (0, T]$. Then there exists a sequence $\{t_m\} \subset E$ such that $t_m \uparrow t$ as $m \to \infty$. Introduce

$$\begin{aligned} I_{ij}(t) :&= \int_{|x|<X} |U_i(x,t) - U_j(x,t)|\,\mathrm{d}x \\ &\leqslant \int_{|x|<X} |U_i(x,t) - U_i(x,t_m)|\mathrm{d}x + \int_{|x|<X} |U_i(x,t_m) - U_j(x,t_m)|\mathrm{d}x + \\ &\quad \int_{|x|<X} |U_j(x,t) - U_j(x,t_m)|\mathrm{d}x \\ &=: I_1 + I_2 + I_3 \ . \end{aligned}$$

It follows directly from the Lebesgue dominated convergence theorem that the middle term $I_2 \to 0$ as $i, j \to \infty$. For $s \in \mathbb{R}$, we denote by $[s]$ the largest integer $\leqslant s$. Since $U_i(x,t)$ is constant for $kh_i \leqslant t < (k+1)h_i$ or $k \leqslant \frac{t}{h_i} < k+1$, we have

$$U_i\,(x,t) = U_i\left(x, \left[\frac{t}{h_i}\right]h_i\right) \ .$$

We use this notation in I_1 (and in I_3):

$$I_1 = \int_{|x|<X} \left| U_i(x, \left[\frac{t}{h_i}\right]h_i) - U_i(x, \left[\frac{t_m}{h_i}\right]h_i) \right| \mathrm{d}x \ .$$

If $[\frac{t}{h_i}] - [\frac{t_m}{h_i}]$ is even, then the integrant in I_1 is constant on intervals of length $2l_i$. We find

$$I_1 \leqslant \sum_{|n| \leqslant \frac{X}{l_i},\ \left(n, [\frac{t}{h_i}]\right) \in S^1} \left| u_n^{[\frac{t}{h_i}]} - u_n^{[\frac{t_m}{h_i}]} \right| 2l_i \ \leqslant\ C^* \left(\left[\frac{t}{h_i}\right] - \left[\frac{t_m}{h_i}\right] \right) h_i \ .$$

Using $s - 1 \leqslant [s] \leqslant s$ gives

$$[s_1] - [s_2] \leqslant s_1 - s_2 + 1 \ .$$

Hence

$$I_1 \leqslant C^*\,(t - t_m + h_i) \ .$$

If $[\frac{t}{h_i}] - [\frac{t_m}{h_i}]$ is odd, then the integrant in I_1 is only constant on intervals of length l_i. Correcting for the differences gives

$$\begin{aligned} I_1 &\leqslant \sum_{|n| \leqslant \frac{X}{l_i},\ \left(n, [\frac{t}{h_i}]\right) \in S^1} \left| u_n^{[\frac{t}{h_i}]} - u_{n+1}^{[\frac{t_m}{h_i}]} \right| 2l_i \ + \sum_{|n| \leqslant \frac{X}{l_i},\ \left(n, [\frac{t}{h_i}]\right) \in S^1} \left| u_n^{[\frac{t}{h_i}]} - u_{n+2}^{[\frac{t}{h_i}]} \right| l_i \\ &\leqslant C^* \left(\left[\frac{t}{h_i}\right] - \left[\frac{t_m}{h_i}\right] \right) h_i + C l_i \leqslant C^*\,(t - t_m) + C^* h_i + \frac{C}{\delta} h_i \ . \end{aligned}$$

Similar expressions are obtained for I_3. Thus we have

$$I_1 + I_3 \ \leqslant\ 2C^*\,(t - t_m) + \left(C^* + \frac{C}{\delta} \right) (h_i + h_j) \ .$$

Now for any $\varepsilon > 0$, we first choose m such that

$$2C^*\,(t - t_m) < \frac{\varepsilon}{2} \ ,$$

which shows that

$$I_{ij}\,(t) < \varepsilon \qquad \text{for } i, j \text{ sufficiently large.}$$

This completes the proof of the proposition. □

Next we show

Proposition 5.13. *The sequence $\{U_i\}_{i=1}^{\infty}$ is a Cauchy sequence in the space $L^1\,([-X, X] \times [0, T])$.*

Proof. We show that for any $\varepsilon > 0$

$$\int_0^T I_{ij}(t)\,\mathrm{d}t < \varepsilon$$

provided i, j are sufficiently large. We first show that

$$I_{ij}(t) \to 0 \qquad \text{as } i, j \to \infty, \text{ uniformly on compacta of } (0, T]$$

i.e. on intervals of the form $0 < \tau \leqslant t \leqslant T$. Let $\tau \in (0, T)$. Then for any $\varepsilon > 0$ choose a finite subset $F \subset E$ such that if $t \in [\tau, T]$ there exists $t_m < t$, $t_m \in F$, satisfying

$$2C^*(t - t_m) < \frac{\varepsilon}{2}\,.$$

Then for i, j sufficiently large we have

$$I_2(t_m) + \left(C^* + \frac{C}{\delta}\right)(h_i + h_j) < \frac{\varepsilon}{2} \qquad \text{for all } t_m \in F\,,$$

which implies that

$$I_{ij}(t) < \varepsilon \quad \text{for all } \tau \leqslant t \leqslant T\,.$$

Next we write

$$\int_0^T \int_{|x|<X} |U_i(x,t) - U_j(x,t)|\mathrm{d}x\mathrm{d}t =$$

$$\int_0^\tau \int_{|x|<X} |U_i(x,t) - U_j(x,t)|\mathrm{d}x\mathrm{d}t + \int_\tau^T \int_{|x|<X} |U_i(x,t) - U_j(x,t)|\mathrm{d}x\mathrm{d}t\,.$$

Then given any $\varepsilon > 0$, first choose τ small such that $8\tau X M < \varepsilon$. This implies that the first integral on the right is less than $\varepsilon/2$. Next choose i, j large so that the second integral is less than $\varepsilon/2$ as well. This completes the proof. □

Since Propositions 5.12 and 5.13 hold for arbitrary $X, T > 0$, we use again a diagonal process and obtain a sequence $\{U_i\}_{i=1}^\infty$, with $h_i \to 0$ as $i \to \infty$, and a measurable function $u \in L^1_{\mathrm{loc}}(\overline{Q})$ so that for $i \to \infty$

(*i*) $U_i \to u$ in $L^1_{\mathrm{loc}}(\overline{Q})$;

(*ii*) $U_i(\cdot, t) \to u(\cdot, t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ for all $t > 0$;

(*iii*) $U_i(x, t) \to u(x, t)$ a.e. in Q.

We immediately have

$$u \in L^\infty(Q) \qquad \text{with } \|u\|_{L^\infty(Q)} \leqslant M = \|u_0\|_{L^\infty(\mathbb{R})}\,.$$

The remaining part of the proof, that is to show that u is indeed a weak solution which satisfies Theorem 5.1, is quite technical and will be discussed here only in general terms. Details can be found in OLEINIK [56] and SMOLLER [67].

The key idea is to write the difference scheme (5.3) in the form

$$\frac{u_n^{k+1} - u_n^k}{h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2l^2}\frac{l^2}{2h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2l} = 0\,.$$

We multiply this expression by the value of the test function at $(x,t)=(nl,kh)$. With $\varphi_n^k=\varphi(nl,kh)$, there results

$$\begin{aligned}\frac{\varphi_n^{k+1}u_n^{k+1}-\varphi_n^k u_n^k}{h} - u_n^{k+1}\frac{\varphi_n^{k+1}-\varphi_n^k}{h} + \frac{l^2}{2h}\frac{2\varphi_n^k-\varphi_{n+1}^k-\varphi_{n-1}^k}{l^2}u_n^k \\ + \frac{\varphi_{n+1}^k u_n^k-\varphi_n^k u_{n-1}^k}{2h} + \frac{\varphi_{n-1}^k u_n^k-\varphi_n^k u_{n+1}^k}{2h} + \frac{\varphi_{n+1}^k f(u_{n+1}^k)-\varphi_{n-1}^k f(u_{n-1}^k)}{2l} \\ - f(u_{n+1}^k)\frac{\varphi_{n+1}^k-\varphi_n^k}{2l} - f(u_{n-1}^k)\frac{\varphi_n^k-\varphi_{n-1}^k}{2l} = 0\ . \qquad (5.18)\end{aligned}$$

Since $\varphi(x,t)=0$ for large $|x|$ and t, we similarly have $\varphi_n^k=0$ for large $|n|$ and k. This allows us to sum (5.18) over all $n\in\mathbb{Z}$ and $k\in\mathbb{Z}^+$. The resulting expression misses the first (except for $k=0$), fourth, fifth and sixth terms because these contributions cancel. After multiplying by hl, one finds

$$\begin{aligned}-l\sum_n u_n^0\varphi_n^0 + hl\Bigg\{\sum_{k,n}\Bigg(-u_n^{k+1}\frac{\varphi_n^{k+1}-\varphi_n^k}{h} - \frac{l^2}{2h}\frac{\varphi_{n+1}^k-2\varphi_n^k+\varphi_{n-1}^k}{l^2}u_n^k\Bigg) \\ - \sum_{k,n} f(u_{n+1}^k)\frac{\varphi_{n+1}^k-\varphi_n^k}{2l} - \sum_{k,n} f(u_{n-1}^k)\frac{\varphi_n^k-\varphi_{n-1}^k}{2l}\Bigg\} = 0\ . \qquad (5.19)\end{aligned}$$

In the definition of the piecewise constant approximations $U_{h,l}(x,t)$ we used points $(n,k)\in S^1$ ($n-k$ even). However, in expression (5.19) we sum over all points (n,k). To use $U_{h,l}$ in (5.19) we take

$$u_n^0 = \begin{cases} u_0(nl) & n \text{ even}\ , \\ u_0\big((n-1)l\big) & n \text{ odd}\ . \end{cases}$$

This implies

$$u_n^k = u_{n-1}^k \quad \text{for } (n,k)\in S^2$$

and

$$U_{h,l}(x,t) = u_n^k \quad \text{for } nl \leqslant x < (n+1)l,\ kh \leqslant t < (k+1)h\ ,$$

now for all (n,k). Using $U_{h,l}$ in (5.19) and replacing the summations by integrations we obtain

$$-\int_{t=0} U_{h,l}\varphi + \delta_1 - \iint_{t\geqslant 0} U_{h,l}\varphi_t + \delta_2 - \frac{l^2}{2h}\iint_{t\geqslant 0} U_{h,l}\varphi_{xx} + \delta_3 - \iint_{t\geqslant 0} f(U_{h,l})\varphi_x + \delta_4 = 0\ ,$$

where $\delta_i\to 0$ as $h,l\to 0$. Replacing $U_{h,l}$ by U_i (see Propositions 5.12 and 5.13) we get

$$\iint_{t\geqslant 0}\Big(U_i\varphi_t + f(U_i)\varphi_x\Big) + \frac{l_i^2}{2h_i}\iint_{t\geqslant 0} U_i\varphi_{xx} + \int_{t=0} U_i\varphi = \delta(h_i,l_i) \qquad (5.20)$$

where $\delta(h_i,l_i)\to 0$ as $i\to\infty$. Note that again $\dfrac{l_i^2}{2h_i}$ appears as numerical dispersion. Using the convergence properties of the sequence $\{U_i\}$ and $\dfrac{l_i^2}{2h_i}=\dfrac{1}{2\delta^2}h_i\to 0$ as $i\to\infty$, we obtain

$$\iint_{t\geqslant 0}\Big(u\varphi_t + f(u)\varphi_x\Big) + \int_{t=0} u_0\varphi = 0\ .$$

Similarly one shows that Proposition 5.6 implies

$$\int_{t\geqslant 0}\Big(u\varphi_x + \frac{E}{t}\varphi\Big) \geqslant 0$$

for all $\varphi \in C_0^\infty(Q)$, $\varphi \geqslant 0$. This is the weak form of entropy inequality (5.1).

6 The non-convex case

We now drop the convexity condition on f and consider the initial value problem

$$\text{(P)}\begin{cases} u_t + (f(u))_x = 0 & \text{in } Q \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathbb{R}\ , \end{cases} \tag{6.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is smooth (e.g. $f\in C^2(\mathbb{R})$), with a finite number of inflection points. Such problems occur in petroleum engineering. The Buckley–Leverett equation without gravity has a monotone convex-concave flux with one inflection point only. Including gravity may result in a non-monotone flux with two inflection points. This is discussed in Chapter 12 and in exercise 8, see Chapter 14.

We shall derive a general entropy condition to distinguish between admissible and non-admissible shocks. Before we proceed we recall that for convex f and piecewise smooth solutions, admissible shocks are those for which

$$u_\mathrm{l} > u_\mathrm{r}\ . \tag{6.2}$$

Here u_l and u_r denote the left and right limit at the shock.

6.1 Travelling waves

We give an argument in terms of travelling waves to capture the viscous profile at shocks. Consider a travelling wave solution of the equation

$$u_t + (f(u))_x = \nu u_{xx} \qquad \text{in } Q$$

where $\nu>0$, which satisfies

$$u(-\infty,t) = u_\mathrm{l}, \qquad u(+\infty,t) = u_\mathrm{r} \qquad \text{for all } t>0\ .$$

Setting

$$u(x,t) = v(\eta) \qquad \text{with } \eta = \frac{x-ct}{\nu}\ ,$$

we obtain the boundary value problem

$$\begin{aligned} -cv' + (f(v))' &= v'' \qquad \text{on } \mathbb{R}\ , \\ v(-\infty) = u_\mathrm{l}\ , &\qquad v(+\infty) = u_\mathrm{r}\ . \end{aligned} \tag{6.3}$$

The differential equation can be integrated to yield

$$-cv + f(v) = v' + A \, .$$

Using $v'(\pm\infty) = 0$ (verify!) we obtain

$$-cu_\mathrm{l} + f(u_\mathrm{l}) = A \qquad \text{and} \qquad -cu_\mathrm{r} + f(u_\mathrm{r}) = A \, .$$

This gives for the travelling wave speed

$$c = \frac{f(u_\mathrm{r}) - f(u_\mathrm{l})}{u_\mathrm{r} - u_\mathrm{l}} \, .$$

Observe that this expression does not depend on ν and coincides with the shockspeed as given by the Rankine-Hugoniot condition (2.2). What remains is the first order equation

$$v' = f(v) - f(u_\mathrm{l}) - c(v - u_\mathrm{l}) \, . \tag{6.4}$$

Now suppose there exists a $\hat{v}$ between u_l and u_r such that

$$f(\hat{v}) - f(u_\mathrm{l}) - c(\hat{v} - u_\mathrm{l}) = 0 \, .$$

Then by a uniqueness argument a solution cannot satisfy both boundary conditions in (6.3). Hence a travelling wave is strictly monotone and

- $u_\mathrm{l} > u_\mathrm{r} \quad \Rightarrow \quad v' < 0 \quad \Rightarrow \quad v < u_\mathrm{l}$,
- $u_\mathrm{l} < u_\mathrm{r} \quad \Rightarrow \quad v' > 0 \quad \Rightarrow \quad v > u_\mathrm{l}$.

In both cases equation (6.4) implies

$$\frac{f(v) - f(u_\mathrm{l})}{v - u_\mathrm{l}} > c = \frac{f(u_\mathrm{r}) - f(u_\mathrm{l})}{u_\mathrm{r} - u_\mathrm{l}} \tag{6.5}$$

for all v between u_l and u_r.

Condition (6.5) is a necessary and sufficient condition for the existence of a travelling wave. The above argument shows that it is necessary. To prove that (6.5) is also sufficient we use (6.5) in (6.4) and integrate

$$\frac{v'}{f(v) - f(u_\mathrm{l}) - c(v - u_\mathrm{l})} = 1 \, ,$$

to obtain

$$\int_{\frac{1}{2}(u_\mathrm{l}+u_\mathrm{r})}^{v(\eta)} \frac{\mathrm{d}s}{f(s) - f(u_\mathrm{l}) - c(s - u_\mathrm{l})} = \eta \, .$$

This defines a unique travelling wave $v = v(\eta)$ which satisfies $v(0) = \frac{1}{2}(u_\mathrm{l} + u_\mathrm{r})$.

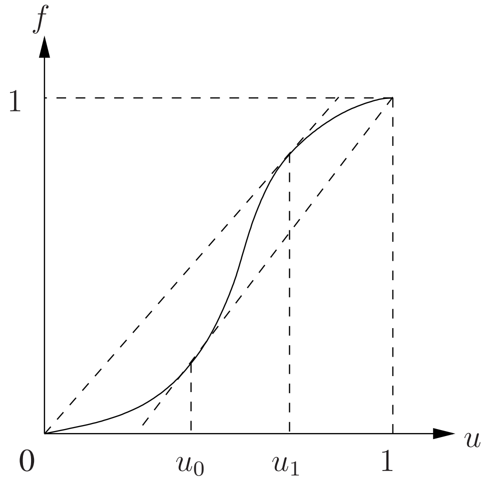


Figure 6.1. *The function f*

Suppose f is convex-concave as in Figure 6.1. What are possible travelling waves?

- $u_l > u_r = 0$: $f(v) < f(u_l) - (u_l - v)\dfrac{f(u_l) - f(u_r)}{u_l - u_r}$. A travelling wave exists up to $0 = u_r < u_l \leqslant u_1$.

- $u_l < u_r = 1$: $f(v) > f(u_l) + (v - u_l)\dfrac{f(u_l) - f(u_r)}{u_l - u_r}$. A travelling wave exists up to $u_0 \leqslant u_l < u_r = 1$.

The numbers u_0 and u_1 are defined in Figure 6.1. Since (6.5) holds for any $\nu > 0$, it can be considered as an additional entropy condition for the corresponding hyperbolic equation. Thus we have:

Entropy condition for piecewise smooth solutions:

$$\text{(E)} \qquad \frac{f(u) - f(u_l)}{u - u_l} \geqslant \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

for all u between u_l and u_r.

Condition (E) ensures the existence of travelling waves $(\nu > 0)$ connecting the levels u_l as $\eta = \dfrac{x - ct}{\nu} \to -\infty$ and u_r as $\eta = \dfrac{x - ct}{\nu} \to +\infty$.

Next we give a result of QUINN [61], which implies that piecewise smooth solutions of (P), which satisfy condition (E) are unique.

Theorem 6.1. *Suppose u and v are piecewise smooth solutions of equation* (6.1) *in Q, with initial data u_0 and v_0 which are piecewise smooth such that $u_0 - v_0 \in L^1(\mathbb{R})$. If u and v satisfy condition* (E) *then*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \leqslant \|u_0 - v_0\|_{L^1(\mathbb{R})}$$

for all $t \geqslant 0$.

6.2 Construction of solutions

Let $f \in C^2([0,1])$ be convex-concave such that

$$\begin{cases} f(0) = 0 \\ f(1) = 1 \\ f'(s) > 0 & \text{for } 0 < s < 1 \\ f''(s) > 0 & \text{for } 0 < s < \tilde{s} \\ f''(s) < 0 & \text{for } \tilde{s} < s < 1 \,, \end{cases}$$

for some $\tilde{s} \in (0,1)$. Consider the Riemann problems

$$\text{(I)} \begin{cases} u_t + (f(u))_x = 0 & \text{in } Q \,, \\ u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \end{cases} \quad \text{and} \quad \text{(II)} \begin{cases} u_t + (f(u))_x = 0 & \text{in } Q \,, \\ u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \,. \end{cases} \end{cases}$$

We construct solutions as combinations of shocks, satisfying (E), and rarefaction waves. They are the unique entropy solutions.

Problem I. Consider the point s_{m} where

$$\frac{f(s_{\mathrm{m}}) - f(0)}{s_{\mathrm{m}} - 0} = \frac{f(s_{\mathrm{m}})}{s_{\mathrm{m}}} = f'(s_{\mathrm{m}}) \,.$$

Note that at s_{m} the shock speed and the rarefaction speed coincide. The solution consists of a constant state ($= 1$) (if $f'(1) > 0$), followed by a rarefaction connecting 1 and s_{m}, followed by a shock connecting s_{m} and 0, see Figure 6.2.

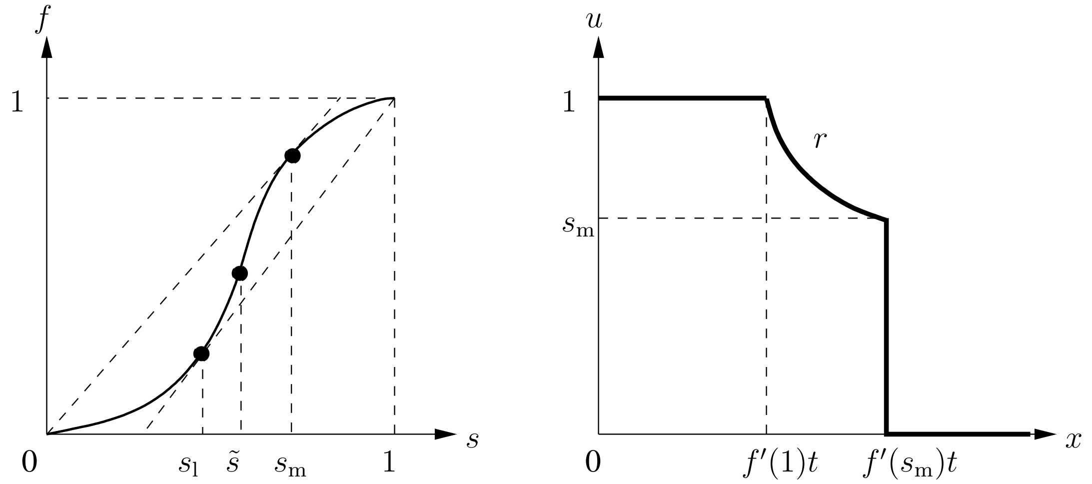


Figure 6.2. *Solution of* **Problem I** *for* $t > 0$

Across the shock the entropy condition is satisfied. Other choices for s_{m} lead to a contradiction:

- $f'(s^1_{\mathrm{m}}) < \dfrac{f(s^1_{\mathrm{m}})}{s^1_{\mathrm{m}}}$ gives a solution that violates (E), see Figure 6.3a;

- $f'(s_{\rm m}^2) > \dfrac{f(s_{\rm m}^2)}{s_{\rm m}^2}$ gives a multivalued solution, see Figure 6.3b.

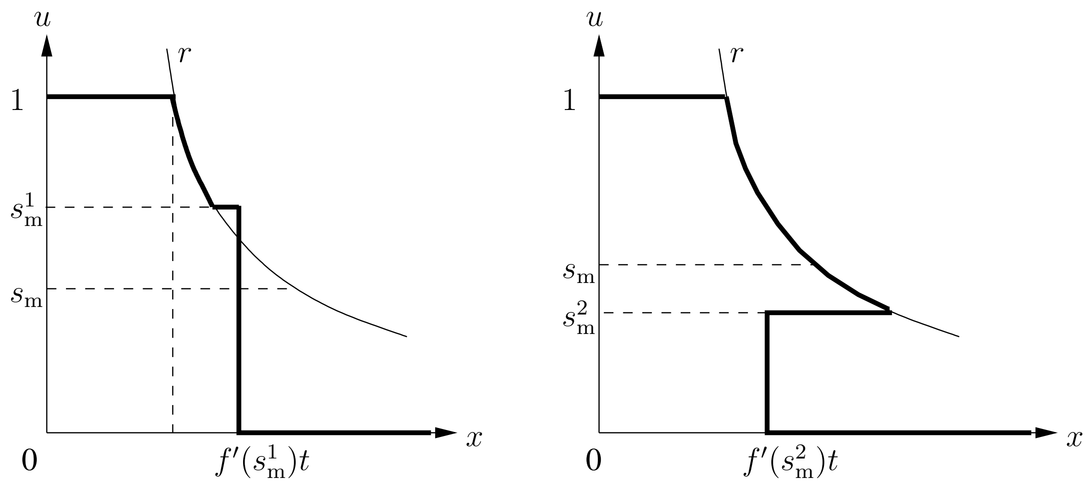


Figure 6.3. *Consequence of other* $s_{\rm m}$

Problem II. Consider the point $s_{\rm l}$ when

$$\frac{f(1)-f(s_{\rm l})}{1-s_{\rm l}} = \frac{1-f(s_{\rm l})}{1-s_{\rm l}} = f'(s_{\rm l}) \, .$$

Then the solution has a constant state ($=0$) (if $f'(0)>0$), followed by a rarefaction connecting 0 and $s_{\rm l}$, followed by a shock connecting $s_{\rm l}$ and 1.

6.3 Weak entropy solutions

Following KRUZKOV [43] we generalize the definition of a weak entropy solution.

Definition 6.2. *A function* $u \in L^\infty(Q)$ *is called a weak entropy solution of* (P) *if*

(*i*) *for all* $k \in \mathbb{R}$

$$\int_Q \big\{ |u-k|\varphi_t + \operatorname{sign}(u-k)\,(f(u)-f(k))\,\varphi_x \big\} \mathrm{d}x\mathrm{d}t \geqslant 0 \, ,$$

for all $\varphi \in C_0^\infty(Q)$ *with* $\varphi \geqslant 0$;

(*ii*) *there exists a set* $\Sigma \subset (0,\infty)$ *with* $\operatorname{meas}(\Sigma)=0$, *such that if* $t \in (0,\infty)\setminus\Sigma$ *the function* $u(x,t)$ *is defined almost everywhere in* $\mathbb{R}$ *and is such that for any* $L>0$

$$\lim_{\substack{t\downarrow 0 \\ t\in(0,\infty)\setminus\Sigma}} \int_{-L}^{L} |u(x,t)-u_0(x)|\mathrm{d}x = 0 \, .$$

Note: since $\varphi\in C_0^\infty(Q)$, the initial condition does not enter the weak form of the equation. Statement (*ii*) is needed to relate a solution to its given value at $t=0$. It means that $||u(t)-u_0||_{L^1_{\mathrm{loc}}(\mathbb{R})}\to 0$ as $t\downarrow 0$, with $t\in(0,\infty)\setminus\Sigma$.

Kruzkov establishes existence (via the viscosity method) and uniqueness of weak entropy solutions. Later CRANDALL & MAJDA [17] presented an existence proof using monotone difference schemes (Lax, Godunov).

Below we show that Kruzkov's formulation unifies all previous properties for piecewise smooth solutions. Let

$$\mathbf{q} = \big(\mathrm{sign}(u-k)\,(f(u)-f(k))\,,|u-k|\big)\ .$$

Then we have

$$\int_Q \mathbf{q}\cdot \mathrm{grad}\,\varphi \geqslant 0$$

for all $\varphi\in C_0^\infty(Q)$ with $\varphi\geqslant 0$ and for all $k\in\mathbb{R}$. For $u\in L^\infty(Q)$ we take $k>\sup u$ and obtain

$$\int_Q (k-u)\varphi_t + (f(k)-f(u))\,\varphi_x \geqslant 0\ .$$

Hence

$$\int_Q u\varphi_t + f(u)\varphi_x \leqslant 0$$

for all $\varphi\in C_0^\infty(Q)$ with $\varphi\geqslant 0$. Similarly we get for $k<\inf u$

$$\int_Q u\varphi_t + f(u)\varphi_x \geqslant 0\ .$$

Thus

$$\int_Q u\varphi_t + f(u)\varphi_x = 0 \qquad \text{for all } \varphi\in C_0^\infty(Q) \text{ with } \varphi\geqslant 0\ ,$$

implying that

$$\int_Q u\varphi_t + f(u)\varphi_x = 0 \qquad \text{for } \textit{all } \varphi\in C_0^\infty(Q)\ .$$

Thus a solution according to Kruzkov's definition is also a weak solution in the original sense. Then we know that (see the results of Chapter 2):

- If u is smooth, it is a classical solution of the equation;
- Across shocks (assuming the solution to be piecewise smooth) the Rankine–Hugoniot condition holds.

Next consider Figure 6.4 in which u is C^1 to the left and right of the shock curve and continuous up to P (from both sides). Let $u_{\mathrm{l}}<u_{\mathrm{r}}$. Then, given any $k\in(u_{\mathrm{l}},u_{\mathrm{r}})$, there exists a disk D_k, centered at P, such that

$$k>u \text{ in } D_k^{\mathrm{l}} \qquad \text{and} \qquad k<u \text{ in } D_k^{\mathrm{r}}\ .$$

This implies for

$$\mathbf{q} = (f(k)-f(u), k-u)$$

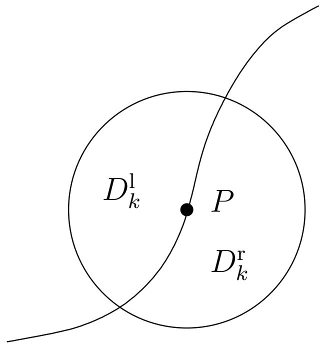


Figure 6.4.

that $\operatorname{div}\mathbf{q}=0$ in D_k^{l} and for

$$\mathbf{q} = (f(u) - f(k), u - k)$$

that $\operatorname{div}\mathbf{q}=0$ in D_k^{r}. In D_k we also have

$$\int_{D_k} \mathbf{q} \cdot \operatorname{grad}\varphi \geqslant 0 \qquad \text{for all } \varphi \in C_0^\infty(D_k) \text{ with } \varphi \geqslant 0 \ .$$

Combining these statements gives

$$\int_{D_k \cap \text{ shock curve}} (\mathbf{q}_{\mathrm{r}} - \mathbf{q}_{\mathrm{l}}) \cdot \mathbf{n}\varphi \geqslant 0 \qquad \text{for all } \varphi \in C_0^\infty(D_k) \text{ with } \varphi \geqslant 0 \ ,$$

and in particular

$$(\mathbf{q}_{\mathrm{r}} - \mathbf{q}_{\mathrm{l}}) \cdot \mathbf{n} \geqslant 0 \qquad \text{at } P \text{ for all } k \in (u_{\mathrm{l}}, u_{\mathrm{r}}) \ .$$

We show that this inequality is equivalent to Oleinik's shock condition. It follows immediately that

$$\tan\,\alpha\,(\mathbf{q}_{\mathrm{r}} - \mathbf{q}_{\mathrm{l}})_t \geqslant (\mathbf{q}_{\mathrm{r}} - \mathbf{q}_{\mathrm{l}})_x \qquad \text{for all } k \in (u_{\mathrm{l}}, u_{\mathrm{r}})$$

and written out

$$\frac{f(u_{\mathrm{r}}) - f(u_{\mathrm{l}})}{u_{\mathrm{r}} - u_{\mathrm{l}}} \{u_{\mathrm{r}} + u_{\mathrm{l}} - 2k\} \geqslant \{f(u_{\mathrm{r}}) + f(u_{\mathrm{l}}) - 2f(k)\} \ .$$

This yields

$$\frac{f(u_{\mathrm{r}}) - f(k)}{u_{\mathrm{r}} - k} \leqslant \frac{f(u_{\mathrm{r}}) - f(u_{\mathrm{l}})}{u_{\mathrm{r}} - u_{\mathrm{l}}} \qquad \text{for all } k \in (u_{\mathrm{l}}, u_{\mathrm{r}}) \ ,$$

which is condition (E).

Below we present a brief motivation concerning Definition 6.2. Kruzkov applied the vanishing viscosity method to establish existence of weak solutions of (P). For $\nu > 0$ he considered the problem

$$\begin{cases} u_t + (f(u))_x = \nu u_{xx} & \text{in } Q \ , \\ u(x,0) = u_0(x) & \text{on } \mathbb{R} \ , \end{cases}$$

and he used a compactness argument in $L^1_{\mathrm{loc}}(Q)$ to obtain convergence as $\nu \to 0$.

Why the special form in Definition 6.2? Let $\psi : \mathrm{I\!R} \to \mathrm{I\!R}$ be a smooth convex function and let $\varphi \in C_0^\infty(Q)$, $\varphi \geqslant 0$. First multiply the viscosity equation by $\psi'(u)$:

$$\psi'(u)u_t - \nu\psi'(u)u_{xx} + \psi'(u)f'(u)u_x = 0 \ .$$

Using

$$(\psi(u))_{xx} = \psi'(u)u_{xx} + \psi''(u)u_x^2$$

we write

$$\psi(u)_t - \nu(\psi(u))_{xx} + \frac{\partial}{\partial x}\int_k^u \psi'(s)f'(s)\mathrm{d}s = -\nu\psi''(u)u_x^2 \leqslant 0 \ .$$

Multiplying by φ and integrating over Q gives

$$\int_Q \left\{ \psi(u)\{\varphi_t + \nu\varphi_{xx}\} + \left(\int_k^u \psi'(s)f'(s)\mathrm{d}s \right) \varphi_x \right\} \mathrm{d}x\mathrm{d}t \geqslant 0 \ .$$

In this inequality we may replace the smooth $\psi(u)$ by $|u-k|$. Then we send $\nu \downarrow 0$ and obtain (*i*).

Part II

Systems

7 Examples

In this chapter we consider a number of different models from physics and engineering. In their mathematical formulation, these models are given in terms of coupled first order partial differential equations. We learn how to construct these equations and we find out what questions to pose when solving them. The derivations will be quite concise. References are given in each example.

7.1 Motion of ideal (perfect) fluid

Consider the one-dimensional flow of an *ideal* or *perfect* fluid. Such a fluid has no internal friction (zero viscosity) and no heat exchange between different parts of the fluid (and the outside world) takes place. Below we present the corresponding transport equations. For an extensive treatment we refer to LANDAU & LIFSCHITZ [44] and VON MISES & FRIEDRICHS [54].

The equation of continuity, or the mass balance equation, is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \, , \tag{7.1}$$

where $\rho = \rho(x,t)$ and $v = v(x,t)$ denote the density and velocity of the fluid, respectively, at a given point x in space and at a given time t.

The equation of motion of a volume element in the fluid results from Newton's second law of dynamics

$$\rho \frac{Dv}{Dt} + \frac{\partial p}{\partial x} = 0 \, , \tag{7.2}$$

in which p denotes the fluid pressure and $\frac{Dv}{Dt}$ the rate of change of the velocity of a given fluid particle as it travels through space. Using in (7.2) the definition

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \tag{7.3}$$

gives

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \, . \tag{7.4}$$

The operator (7.3) is the *material derivative* and (7.4) is known as *Euler's equation.* Combining (7.4) and (7.1) gives

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 + p) = 0 \ .$$

The absence of heat exchange between different fluid particles means that the flow of an ideal fluid is *adiabatic.* This implies that for any given fluid particle the *entropy* S remains constant as the particle moves through space. Hence

$$\frac{DS}{Dt} = 0 \qquad\qquad \text{(adiabatic flow)}$$

or

$$\frac{\partial S}{\partial t} + v\frac{\partial S}{\partial x} = 0 \ .$$

Combining this equation with (7.1), results in the *continuity equation for the entropy*

$$\frac{\partial \rho S}{\partial t} + \frac{\partial}{\partial x}(\rho v S) = 0 \ ,$$

where $\rho v S$ denotes the entropy density flux.

Hence the space-time behaviour of the state (ρ, v and S or ρ, v and p) of an ideal fluid in a one-dimensional setting is described by the three coupled differential equations

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \qquad &= 0 \ , \\
\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 + p) &= 0 \ , \\
\frac{\partial \rho S}{\partial t} + \frac{\partial}{\partial x}(\rho v S) \quad\ &= 0 \ .
\end{aligned}$$

To solve this system one needs an additional algebraic relation bewteen ρ, p and S. For example in the case of an *ideal gas* it can be shown that

$$S = c_v \log(p/\rho^\gamma) + \text{constant} \ , \tag{7.5}$$

where $\gamma = c_p/c_v > 1$ and c_p and c_v are the specific heat at constant pressure and volume, respectively: see FEYNMAN [24] for a clear discussion on this subject and also LEVEQUE [47]. Instead of (7.5) one often writes

$$p = \kappa \exp\{S/c_v\}\, \rho^\gamma \ . \tag{7.6}$$

When the entropy remains constant throughout the fluid and during its motion, we call the flow *isentropic.* Hence the equations for *isentropic gas* flow are given by

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0 \ , \\[2ex] \dfrac{\partial \rho v}{\partial t} + \dfrac{\partial}{\partial x}(\rho v^2 + \overline{\kappa}\rho^\gamma) = 0 \ , \end{cases} \tag{7.7}$$

for the unknowns ρ and v. For $\gamma = 1$, these equations describe isothermal flow. Thus we call (7.7) with $\gamma = 1$, the *isothermal gas* flow equations.

7.2 Shallow water equations

Consider the movement of water in a channel of uniform width. Let the depth be small compared to the characteristic dimensions of the problem. This allows for the *hydraulic approach* (see for instance [44] and [47]), in which the vertical water velocity can be disregarded with respect to the horizontal water velocity v, which in turn is assumed to be constant in each cross-section: $v = v(x,t)$ only.

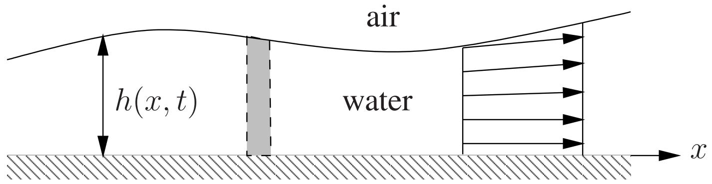


Figure 7.1. *Shallow water approximation*

Let ρ denote the constant density of the water. Mass conservation, applied to a thin vertical layer of water (the shaded column in Figure 7.1) gives

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0 \tag{7.8}$$

and momentum conservation yields

$$\rho h \frac{Dv}{Dt} + \frac{\partial}{\partial x}\left\{\int_0^h \rho g(h-z)\,\mathrm{d}z\right\} = 0\ .$$

This results in the Euler equation

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left\{\frac{v^2}{2} + gh\right\} = 0\ . \tag{7.9}$$

Putting $z = gh$ in (7.8) and (7.9), we obtain

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial}{\partial x}(vz) = 0\ , \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{v^2}{2} + z\right) = 0\ . \end{cases} \tag{7.10}$$

This system is known as the *shallow water equations*.

7.3 p-System: nonlinear wave equation

We want to rewrite the equations for isentropic flow (7.7) in terms of material or Lagrangian coordinates. In doing this we follow [44]. For a given fluid particle, let $a \in \mathbb{R}$ denote its initial position. The position at time $t > 0$ is found by solving the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v(x,t)\ , \quad t > 0\ , \\[2ex] x(0) = a\ . \end{cases}$$

Denoting the solution by $x = x(a;t)$, we introduce

$$\rho(x,t) = \rho(x(a,t),t) =: \widehat{\rho}(a,t) \; ,$$
$$v(x,t) = v(x(a,t),t) =: \widehat{v}(a,t) \; ,$$

and

$$p(x,t) = p(x(a,t),t) =: \widehat{p}(a,t) \; .$$

The rules for differentiation are

$$\frac{\partial \widehat{\rho}}{\partial t} = v\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} \quad \text{and} \quad \frac{\partial \widehat{\rho}}{\partial a} = \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial a} \; .$$

Tracking fluid particles as in Figure 7.2, we obtain from mass-conservation

$$\int_{a_1}^{a_2} \rho_0(x)\mathrm{d}x = \int_{x(a_1;t)}^{x(a_2;t)} \rho(x,t)\mathrm{d}x \; ,$$

for every $-\infty < a_1 < a_2 < \infty$ and for all $t > 0$.

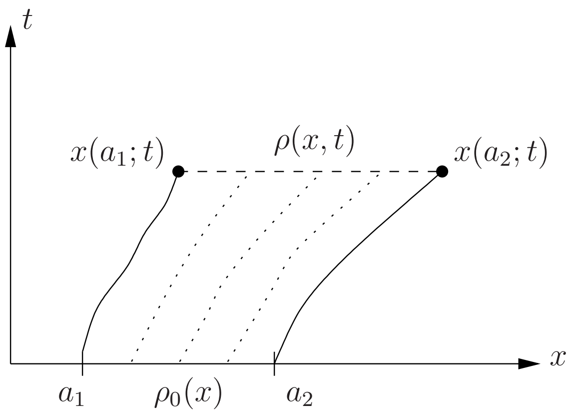


Figure 7.2. *Particle flow in $x - t$ plane*

This implies

$$\frac{\partial x}{\partial a} = \frac{\rho_0}{\rho} = \frac{\rho_0}{\widehat{\rho}} \; .$$

Hence the equations for isentropic flow transform into

$$\frac{\partial \widehat{\rho}}{\partial t} + \frac{\widehat{\rho}^2}{\rho_0}\frac{\partial \widehat{v}}{\partial a} = 0 \quad \text{and} \quad \frac{\partial \widehat{v}}{\partial t} + \frac{1}{\rho_0}\frac{\partial \widehat{p}}{\partial a} = 0 \; .$$

Introducing $\widehat{u} = \dfrac{1}{\widehat{\rho}}$ as the specific volume, and the stretching

$$y(a) := \int^a \rho_0(x)\mathrm{d}x, \quad (\rho_0 \text{ initial density}) \; ,$$

we obtain (after dropping $\widehat{\ }$)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial v}{\partial y} = 0 \ , \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial y} = 0 \ , \end{cases} \tag{7.11}$$

where $p = p(u)$ follows from (7.6): $p = \overline{\kappa} u^{-\gamma}$.

In a more general context, i.e. $p : \mathbb{R}^+ \to \mathbb{R}^+$ such that $p' < 0$, $p'' > 0$, we call (7.11) the p-system. It is studied in detail in SMOLLER [67], where further references are given.

Remark 7.1. *By cross-differentiating equations* (7.11) *we find the nonlinear wave equation*

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 p(u)}{\partial y^2} = 0 \ . \tag{7.12}$$

Remark 7.2. *Considering the first equation in* (7.11) *as a divergence, we know that there exists a stream function ψ, on simply connected domains, such that*

$$\frac{\partial \psi}{\partial y} = u \quad \textit{and} \quad \frac{\partial \psi}{\partial t} = v \ . \tag{7.13}$$

Using the first expression in (7.13) *gives the nonlinear wave equation*

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial}{\partial y} p \left(\frac{\partial \psi}{\partial y} \right) = \frac{\partial^2 \psi}{\partial t^2} + p' \left(\frac{\partial \psi}{\partial y} \right) \frac{\partial^2 \psi}{\partial y^2} = 0 \ . \tag{7.14}$$

Here $\sqrt{-p'}$ denotes the wave speed.

7.4 Chromatography of two solutes

This example is taken from RHEE, ARIS & AMUNDSON [63]. Suppose two solutes A_1 and A_2 are present in a fluid which moves through a homogeneous porous column.

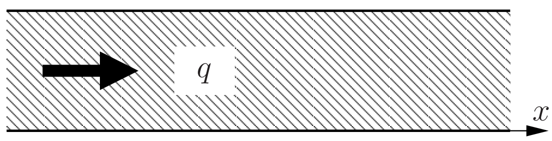


Figure 7.3. *Solute transport through a column*

We suppose that the solute concentrations c_1 and c_2 are at tracer level, which means that the fluid flow is not affected by these concentrations and can be regarded as given $(q > 0)$. The solutes undergo adsorption reactions with the porous skeleton and it is assumed that chemical equilibrium has been reached. If n_1 and n_2 denote the adsorbed concentrations of A_1 and A_2, we find in the absence of dispersion and diffusion the balance equations

$$\frac{\partial}{\partial t} \left(\phi c_1 + (1 - \phi) n_1 \right) + q \frac{\partial c_1}{\partial x} = 0 \tag{7.15a}$$

and

$$\frac{\partial}{\partial t}\left(\phi c_2 + (1-\phi)n_2\right) + q\frac{\partial c_2}{\partial x} = 0 \ . \tag{7.15b}$$

Here $\phi \in (0,1)$ denotes the porosity of the medium. To find n_1 and n_2 we argue as follows. Let the total number of sites at which adsorption takes place be bounded. Then the total adsorbed concentration n_1+n_2 cannot exceed an upperbound, N say. Hence $N-n_1-n_2$ is proportional to the number of vacant sites. For the rate of adsorption of A_1 we have

$$r_{\mathrm{a}_1} = k_{\mathrm{a}_1}(N - n_1 - n_2)c_1 \ .$$

At equilibrium this is balanced by the rate of desorption $r_{\mathrm{d}_1} = k_{\mathrm{d}_1} n_1$, which gives

$$K_1(N - n_1 - n_2)c_1 = n_1 \ , \tag{7.16}$$

where $K_1 = k_{\mathrm{a}_1}/k_{\mathrm{d}_1}$. Similarly we find for A_2

$$K_2(N - n_1 - n_2)c_2 = n_2 \ . \tag{7.17}$$

Equations (7.16) and (7.17) can be solved to give the Langmuir isotherms

$$n_1 = \frac{NK_1c_1}{1 + K_1c_1 + K_2c_2} \quad \text{and} \quad n_2 = \frac{NK_2c_2}{1 + K_1c_1 + K_2c_2} \ . \tag{7.18}$$

Substitution of these expressions into equations (7.15a) and (7.15b) leads to two coupled first order equations for the concentrations c_1 and c_2. Introducing the moving coordinates

$$y = \frac{1-\phi}{q}x \quad \text{and} \quad \tau = t - \frac{\phi}{q}x \ ,$$

equations (7.15a) and (7.15b) become

$$\frac{\partial}{\partial \tau}n_1(c_1, c_2) + \frac{\partial c_1}{\partial y} = 0 \ , \tag{7.19a}$$

$$\frac{\partial}{\partial \tau}n_2(c_1, c_2) + \frac{\partial c_2}{\partial y} = 0 \ . \tag{7.19b}$$

Note that (7.16) and (7.17) also imply

$$c_1 = \frac{1}{K_1}\frac{n_1}{N - n_1 - n_2} \quad \text{and} \quad c_2 = \frac{1}{K_2}\frac{n_2}{N - n_1 - n_2} \ ,$$

which puts (7.19) in the “standard” form

$$\frac{\partial n_1}{\partial \tau} + \frac{\partial}{\partial y}c_1(n_1, n_2) = 0 \ ,$$

$$\frac{\partial n_2}{\partial \tau} + \frac{\partial}{\partial y}c_2(n_1, n_2) = 0 \ .$$

7.5 Polymer flooding

Polymer flooding occurs in reservoir engineering. Polymers are dissolved in the water phase in order to increase the water viscosity and thus to stabilize the water-oil displacement process. This implies that the fractional flow function (F_{w} in Chapter 12) now depends on both the water saturation s and the polymer concentration c. Moreover it is assumed that the polymer undergoes equilibrium adsorption with the immobile phase, say according to a Langmuir isotherm (see also (7.18)). After appropriate scaling and balancing one finds for the water saturation and the polymer concentration the system of coupled equations, see [63] for details,

$$\frac{\partial s}{\partial t} + \frac{\partial f_{\mathrm{w}}}{\partial x} = 0 \tag{7.20a}$$

and

$$\frac{\partial}{\partial t}(sc + c_{\mathrm{a}}) + \frac{\partial}{\partial x}(cf_{\mathrm{w}}) = 0 \ . \tag{7.20b}$$

Here $f_{\mathrm{w}} = f_{\mathrm{w}}(s,c)$ and $c_{\mathrm{a}} = g(c)$. Typical examples are

$$f_{\mathrm{w}}(s,c) = \frac{s^3}{s^3 + \alpha(1+\beta c)(1-s)^2(1+2s)}$$

and

$$g(c) = \frac{NKc}{1+Kc} \ ,$$

where α, β, K are positive constants, and where $0 \leqslant s \leqslant 1$ and $c \geqslant 0$.

7.6 Conclusions

In the previous sections we have derived examples of systems of first order partial differential equations, mostly two equations with two unknowns, arising in different areas of physics and engineering. We also gave references in which the underlying models are treated in great detail and depth.

Up to simple transformations, these systems all have the form of coupled conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \ , \tag{7.21}$$

where $\mathbf{u} : \mathrm{I\!R} \times \mathrm{I\!R}^+ \to \mathrm{I\!R}^n$ and $\mathbf{f} : \mathrm{I\!R}^n \to \mathrm{I\!R}^n$. Furthermore in the examples the vectorfunctions $\mathbf{f}$ are smooth, i.e. the components $f_i = f_i(u_1, u_2, \cdots, u_n), i = 1, 2, \cdots, n$, are smooth functions of the variables $u_1, u_2, \cdots, u_n$.

This motivates us to study in the next chapters systems of the form (7.21) with smooth nonlinearities. In particular Chapter 9 is devoted to the construction of a particular solution of the isothermal gas flow equations: i.e. equations (7.7) with $\gamma = 1$.

8 Linear hyperbolic systems

We consider here the initial value problem for the linear system

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0 \qquad \text{in } Q = \mathbb{R} \times \mathbb{R}^+ \ , \tag{8.1a}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot) \qquad \text{on } \mathbb{R} \ , \tag{8.1b}$$

where $\mathbf{u} : Q \to \mathbb{R}^n$ and A is a constant $n \times n$-matrix. We first verify the well-posedness of this problem. We do this by considering solutions in the form of complex plane waves

$$\mathbf{u}(x, t) = \boldsymbol{\xi} \exp\{i(\lambda t - \mu x)\} \ , \qquad \text{for } (x, t) \in Q \ , \tag{8.2}$$

where $\boldsymbol{\xi}$ is a constant n-vector. At $t = 0$ we want to have a bounded initial value. This forces μ to be real. Substitution of (8.2) into (8.1a) gives

$$\mu A \boldsymbol{\xi} = \lambda \boldsymbol{\xi} \ ,$$

implying that λ/μ is an eigenvalue of the matrix A. Writing $\lambda/\mu = a + bi$ and $\xi_k = |\xi_k| \exp\{i\alpha_k\}$ for each of the n components of the eigenvector $\boldsymbol{\xi}$, we obtain from (8.2) the real valued plane waves

$$u_k(x, t) = |\xi_k| \cos(\alpha_k + \mu(at - x)) \exp\{-\mu bt\} \ , \qquad \text{for } (x, t) \in Q \ ,$$

for $k = 1, 2, \cdots, n$. Now let $N > 0$ be given. Then taking $|\xi_k| = N^{-1}$ and $\mu = -2N \log N/b$, we have

$$\max_{x \in \mathbb{R}} |u_k(x, 0)| = |\xi_k| = N^{-1} \ ,$$

while

$$\max_{x \in \mathbb{R}} \left| u_k(x, N^{-1}) \right| = |\xi_k| \exp\{-\mu b/N\} = N \ .$$

To have continuous dependence on initial data, we must require $b = 0$. This forces A to have real eigenvalues, for which we call system (8.1a) *weakly hyperbolic*. If in addition the eigenvalues are distinct, we say that (8.1a) is a *hyperbolic* system. Throughout this section we assume that (8.1a) is hyperbolic.

8.1 Decoupling of equations

The matrix A can be diagonalized because it has distinct eigenvalues, see for instance STRANG [69]. Let

$$A = T\Lambda T^{-1} ,$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the diagonal matrix of eigenvalues and $T = [\mathbf{t}_1|\mathbf{t}_2|\cdots|\mathbf{t}_n]$ the matrix of linear independent eigenvectors such that

$$A\mathbf{t}_k = \lambda_k \mathbf{t}_k \qquad \text{for } k = 1, 2, \cdots, n .$$

We now solve (8.1a) by making the change of variables $\mathbf{u} = T\mathbf{v}$. Substituting this into (8.1a) and multiplying the result by the constant matrix T^{-1} yields

$$\frac{\partial \mathbf{v}}{\partial t} + \Lambda \frac{\partial \mathbf{v}}{\partial x} = 0 .$$

In other words we have achieved a decoupling of the system (8.1a) into n dependent scalar equations

$$\frac{\partial v_k}{\partial t} + \lambda_k \frac{\partial v_k}{\partial x} = 0 , \qquad k = 1, 2, \cdots, n .$$

Each of these is a constant coefficient linear equation, with solution

$$v_k(x,t) = v_k(x - \lambda_k t, 0) .$$

Since $\mathbf{v} = T^{-1}\mathbf{u}$, the initial value for v_k is simply the k^{th} component of the vector $T^{-1}\mathbf{u}_0$. Thus

$$v_k(x,t) = \big(T^{-1}\mathbf{u}_0(x - \lambda_k t)\big)_k \qquad \text{for } k = 1, 2, \cdots, n . \tag{8.3}$$

Finally we use $\mathbf{u} = T\mathbf{v}$ to obtain as a solution of (8.1)

$$\mathbf{u}(x,t) = \sum_{k=1}^{n} \big(T^{-1}\mathbf{u}_0(x - \lambda_k t)\big)_k \mathbf{t}_k , \quad \text{with } (x,t) \in Q . \tag{8.4}$$

Note that $\mathbf{u}(x,t)$ depends only on the initial data at the points $x - \lambda_k t$. We say that the *domain of dependence* for an arbitrary point $(x^*, t^*) \in Q$ is given by

$$\mathcal{D}(x^*, t^*) = \{x \in \mathbb{R} : x = x^* - \lambda_k t^*, k = 1, 2, \cdots, n\} . \tag{8.5}$$

Curves $x = x_0 + \lambda_k t$, satisfying $x'(t) = \lambda_k$, are called "the characteristics of the k^{th} family", or simply the "k^{th}-characteristics". Note that n distinct characteristics curves pass through each point in the x–t plane.

Remark 8.1. *The domain of dependence in parabolic problems is the whole space on which the problem is defined. For instance, consider the initial value problem for the heat equation* $(d > 0)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} & \text{in } Q , \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R} . \end{cases}$$

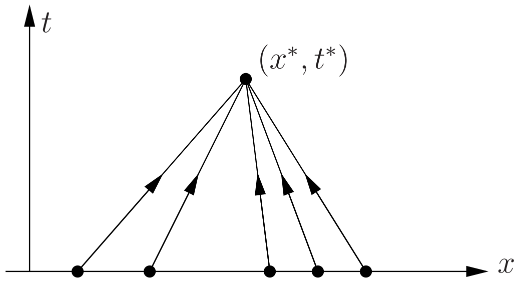


Figure 8.1. *Domain of dependence* (8.5) *for a given point* (x^*, t^*)

The solution is given by

$$\frac{1}{2\sqrt{d\pi t}} \int_{\mathbb{R}} u_0(y) \exp\left\{-(x-y)^2/4dt\right\} \mathrm{d}y \ ,$$

which shows that

$$\mathcal{D}(x^*, t^*) = \mathbb{R} \qquad \textit{for all } (x^*, t^*) \in Q \ .$$

Remark 8.2. *The solution procedure described in this section only applies for piecewise smooth initial data. Across any discontinuity the equation must be satisfied in the sense of the Rankine–Hugoniot shock conditions. We return to this point in the next section where we treat the Riemann problem in detail.*

Application : linear wave equation. Consider the initial value problem for the linear wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{in } Q \ , \\ u(\cdot, 0) = \phi(\cdot) & \text{on } \mathbb{R} \ , \\ \dfrac{\partial u}{\partial t}(\cdot, 0) = \psi(\cdot) & \text{on } \mathbb{R} \ . \end{cases} \tag{8.6}$$

To solve this problem we put the differential equation in (8.6) in the form of a first order hyperbolic system. This is done by introducing the vector

$$\mathbf{u} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial t} \end{pmatrix}$$

and by writing

$$\frac{\partial \mathbf{u}}{\partial t} + A \frac{\partial \mathbf{u}}{\partial x} = 0 \qquad \text{in } Q \ ,$$

with

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \ .$$

The initial data becomes

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot) = \begin{pmatrix} \phi'(\cdot) \\ \\ \psi(\cdot) \end{pmatrix} .$$

The matrix A has eigenvalues $\lambda_1 = -c, \lambda_2 = c$ and the corresponding matrix of eigenvectors is

$$T = \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix} \quad \text{with} \quad T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & c^{-1} \\ 1 & -c^{-1} \end{pmatrix} .$$

Hence according to (8.3) we have

$$\begin{aligned} v_1 &= \left(T^{-1}\mathbf{u}_0(x+ct)\right)_1 = \frac{1}{2}\left\{\phi'(x+ct) + \frac{1}{c}\psi(x+ct)\right\} , \\ v_2 &= \left(T^{-1}\mathbf{u}_0(x-ct)\right)_2 = \frac{1}{2}\left\{\phi'(x-ct) - \frac{1}{c}\psi(x+ct)\right\} , \end{aligned}$$

and from (8.4)

$$\mathbf{u}(x,t) = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \\ \dfrac{\partial u}{\partial t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi'(x+ct) + \dfrac{1}{c}\psi(x+ct) + \phi'(x-ct) - \dfrac{1}{c}\psi(x-ct) \\ \\ c\phi'(x+ct) + \psi(x+ct) - c\phi'(x-ct) + \psi(x-ct) \end{pmatrix} .$$

Finally, integration yields

$$u(x,t) = \frac{1}{2}\left\{\phi(x+ct) + \phi(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\mathrm{d}s .$$

8.2 Riemann problem

The Riemann problem for equation (8.1a) is the initial value problem with piecewise constant data, i.e.

$$\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_\mathrm{l}, & \text{for } x < 0 , \\ \mathbf{u}_\mathrm{r}, & \text{for } x > 0 . \end{cases}$$

Because we are dealing here with a constant coefficient linear system, the Riemann problem can be solved by decoupling the equations. Below we give the construction.

Let us assume without loss of generality that the eigenvalues of the matrix A are ordered:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n .$$

We now proceed as follows. By $\mathbf{u} = T\mathbf{v}$ we have

$$\mathbf{u} = \sum_{k=1}^{n} v_k \mathbf{t}_k ,$$

and thus we start by decomposing $\mathbf{u}_0$ accordingly:

$$\mathbf{u}_\mathrm{l} = \sum_{k=1}^{n} \alpha_k \mathbf{t}_k \quad \text{and} \quad \mathbf{u}_\mathrm{r} = \sum_{k=1}^{n} \beta_k \mathbf{t}_k \ . \tag{8.7}$$

Hence

$$v_k(x,0) = \begin{cases} \alpha_k, & \text{for } x < 0 \ , \\ \beta_k, & \text{for } x > 0 \ , \end{cases}$$

and consequently

$$v_k(x,t) = \begin{cases} \alpha_k & \text{for } x - \lambda_k t < 0 \ , \\ \beta_k & \text{for } x - \lambda_k t > 0 \ . \end{cases}$$

For a given $(x,t) \in Q$, let $K(x,t)$ denote the maximum value of k for which $x - \lambda_k t > 0$. If $x - \lambda_k t < 0$ for all k, we set $K(x,t) = 0$. Then there results

$$\mathbf{u}(x,t) = \sum_{k=1}^{K(x,t)} \beta_k \mathbf{t}_k + \sum_{k=K(x,t)+1}^{n} \alpha_k \mathbf{t}_k \ . \tag{8.8}$$

Note that u is constant for those combinations of x and t giving rise to the same K. This is the case for points taken from cones in the $x-t$ plane. For instance, see also Figure 8.2,

$$K(x,t) = l \quad \text{on} \quad \{(x,t) : t > 0 \text{ and } \lambda_l t < x < \lambda_{l+1} t\} \ .$$

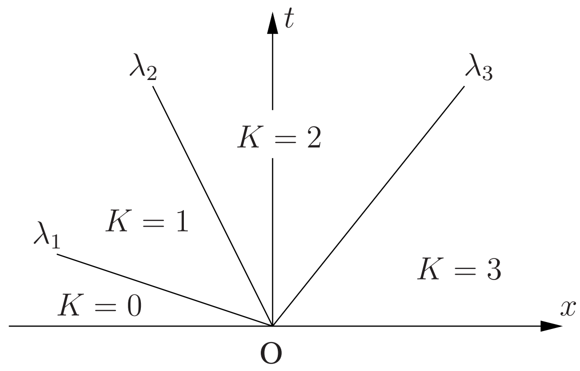


Figure 8.2. *Distribution of K values in the $x-t$ plane*

The example sketched in Figure 8.2 leads to a solution as shown in Figure 8.3. Note that here $n = 3$. Along the k^{th}-characteristic the solution is discontinuous. The jump $[\mathbf{u}]$ is given by

$$[\mathbf{u}] = (\beta_k - \alpha_k)\mathbf{t}_k \ .$$

Note that $\mathbf{f}(\mathbf{u}) = A\mathbf{u}$ jumps according to

$$\begin{aligned} [\mathbf{f}] = A[\mathbf{u}] &= (\beta_k - \alpha_k) A \mathbf{t}_k \\ &= (\beta_k - \alpha_k) \lambda_k \mathbf{t}_k \ . \end{aligned}$$

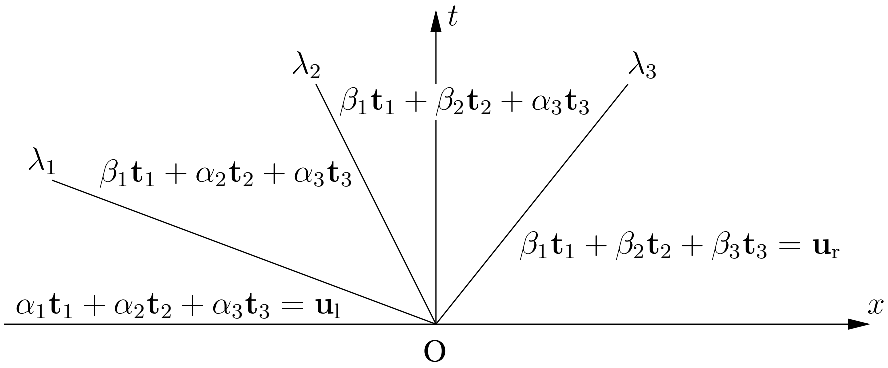


Figure 8.3. *The solution* (8.8) *in the* $x-t$ *plane*

Hence accross the k^{th}-characteristic we have

$$[\mathbf{f}] = \lambda_k[\mathbf{u}] \ ,$$

in which we recognize the Rankine–Hugoniot condition.

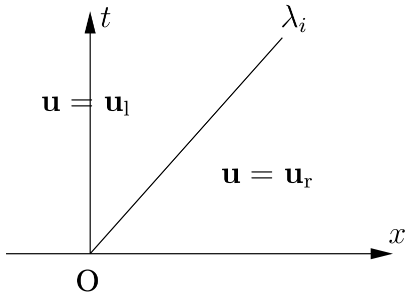


Figure 8.4. *Propagation of a single shock*

An alternative form for the solution $\mathbf{u}$ can be obtained from Figure 8.3. One has

$$\begin{aligned}\mathbf{u}(x,t) &= \mathbf{u}_\mathrm{l} + \sum_{\lambda_k<x/t} (\beta_k - \alpha_k)\mathbf{t}_k \\ &= \mathbf{u}_\mathrm{r} - \sum_{\lambda_k>x/t} (\beta_k - \alpha_k)\mathbf{t}_k \ .\end{aligned}$$

Substracting the equations in (8.7) yields

$$\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l} = \sum_{k=1}^{n} (\beta_k - \alpha_k)\mathbf{t}_k \ .$$

This means that if $\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l}$ is already an eigenvector of A, i.e. $\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l} = c\,\mathbf{t}_i$ for some $i \in \{1, 2, \cdots, n\}$, then $\beta_i - \alpha_i = c$ and $\beta_k - \alpha_k = 0$ for all $k \neq i$. Hence a situation as in Figure 8.4 results. In general, however, the initial jump $\mathbf{u}_\mathrm{l} - \mathbf{u}_\mathrm{r}$ will break up into a sum of jumps (at most n) and each jump $(\beta_k - \alpha_k)\mathbf{t}_k$ will travel with speed λ_k.

When $n=2$, it is often convenient to investigate the solution of Riemann problems in the phase plane, i.e. the u_1-u_2 plane. Each vector is represented by a point in this plane. Furthermore a discontinuity with left and right states $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$, respectively, can only propagate as a single discontinuity if $\mathbf{u}_\mathrm{r}-\mathbf{u}_\mathrm{l}$ is parallel to one of the eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$. Thus given a state $\mathbf{u}_\mathrm{l}$, we find all other states $\mathbf{u}_\mathrm{r}$, for which $\mathbf{u}_\mathrm{r}-\mathbf{u}_\mathrm{l}$ travels as single discontinuity, by taking $\mathbf{u}_\mathrm{r}$ from lines parallel to $\mathbf{t}_1$ and $\mathbf{t}_2$, passing through the point $\mathbf{u}_\mathrm{l}$.

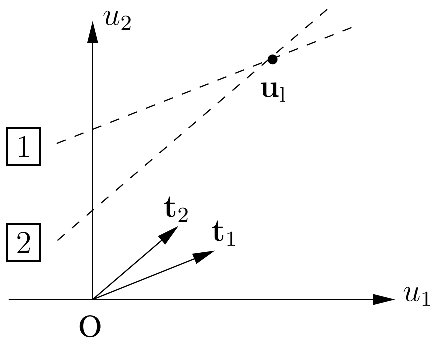


Figure 8.5. *The Hugoniot locus of the state* $\mathbf{u}_\mathrm{l}$

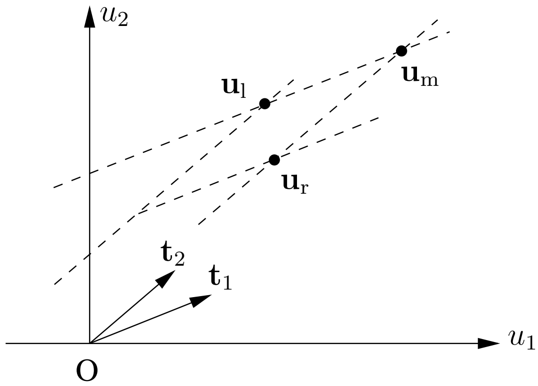


Figure 8.6. *Construction of intermediate state* $\mathbf{u}_\mathrm{m}$

If $\mathbf{u}_\mathrm{r}$ is taken from line $\boxed{1}$ we call the solution a 1-wave. With $\mathbf{u}_\mathrm{r}$ from line $\boxed{2}$ there results a 2-wave. For a general Riemann problem with arbitrary $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$, the solution has two discontinuities travelling with speeds λ_1 and λ_2, with an intermediate state $\mathbf{u}_\mathrm{m}$ given by

$$\mathbf{u}_\mathrm{m} = \beta_1 \mathbf{t}_1 + \alpha_2 \mathbf{t}_2 \; ,$$

so that $\mathbf{u}_\mathrm{l}-\mathbf{u}_\mathrm{m} = (\alpha_1-\beta_1)\mathbf{t}_1$ and $\mathbf{u}_\mathrm{r}-\mathbf{u}_\mathrm{m} = (\beta_2-\alpha_2)\mathbf{t}_2$. This leads to a phase plane picture as in Figure 8.6.

9 Riemann problem for nonlinear equations: the construction

We now consider the nonlinear Riemann problem

$$
\text{(R)}\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \dfrac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \qquad \text{in } Q\ , \\[2ex] \mathbf{u}(x,0) = \begin{cases} \mathbf{u}_\mathrm{l} & \text{for } x < 0\ , \\ \mathbf{u}_\mathrm{r} & \text{for } x > 0\ , \end{cases} \end{cases} \tag{9.1}
$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector valued function. In this chapter the emphasis will be on the *construction* of solutions. In Chapter 10 we study the underlying mathematical framework. The system in (9.1) can be put in the quasilinear form

$$
\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{in } Q\ ,
$$

where $A(\mathbf{u}) = D\mathbf{f}(\mathbf{u})$ is the $n \times n$ Jacobian matrix. Again we call the system hyperbolic if $A(\mathbf{u})$ has real and distinct eigenvalues for all $\mathbf{u}$, at least for all $\mathbf{u}$ in the range where the solution is known to lie, such that $\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_n(\mathbf{u})$. The corresponding n linearly independent eigenvectors are denoted by $\mathbf{t}_k(\mathbf{u})$ for $k = 1, 2, \cdots, n$.

As in the linear case we can find the characteristics by integrating the eigenvalues of $A(\mathbf{u})$. Again there are n distinct characteristic curves passing through each point in Q. For instance the k^{th} characteristic through the point $(x_0, t_0) \in Q$ is determined by the initial value problem

$$
\begin{cases} x'(t) = \lambda_k\big(\mathbf{u}(x(t),t)\big) \\ x(t_0) = x_0\ . \end{cases}
$$

However since λ_k depends on $\mathbf{u}$, the a-priori unknown solution of the problem, we can no longer solve the system by first determining the characteristics and then applying a decoupling.

In the linear case the solution of the Riemann problem consists of n waves (or shocks), which are discontinuities travelling at the characteristic velocities of the system. However, in the nonlinear case the physically relevant solution may contain rarefaction waves as well as shock waves. In the first section we shall disregard the rarefaction waves (and thus the entropy conditions) and construct solutions of the Riemann problem consisting of (n) shocks only, propagating with constant speeds $s_1 < s_2 < \cdots < s_n$. According to the general theory, as demonstrated in Chapter 10, this can always be done if $||\mathbf{u}_\mathrm{l} - \mathbf{u}_\mathrm{r}||$ is sufficiently small. After this construction we discuss the entropy conditions and build rarefaction waves into the solution.

9.1 Shocks

A shock or discontinuity in a solution of equation (9.1) is characterized by the values $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ on the left and right side, respectively, and by the speed s. We shall use the notation $\{\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}, s\}$ for a shock. Interpreting solutions in a weak sense, as in the scalar case, we obtain that any shock should satify the Rankine–Hugoniot relation

$$\mathbf{f}(\mathbf{u}_\mathrm{r}) - \mathbf{f}(\mathbf{u}_\mathrm{l}) = s(\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l}) \ . \tag{9.2}$$

Suppose we fix a point $\mathbf{u}^0 \in \mathbb{R}^n$ and attempt to determine the set of all points $\mathbf{u} \in \mathbb{R}^n$ that can be connected to $\mathbf{u}^0$ by a discontinuity satisfying (9.2) for some s. In the linear case, this leads to n families of solutions $\mathbf{u}_n(\xi)$, $-\infty < \xi < \infty$, each with its own shockspeed s_n: i.e. for $1 \leqslant k \leqslant n$

$$\begin{cases} \mathbf{u}_k = \mathbf{u}_k(\xi; \mathbf{u}^0) = \mathbf{u}^0 + \xi \mathbf{t}_k \ , \\ s_k = \lambda_k \ . \end{cases}$$

See also Figure 8.5 (for the case $n = 2$).

In the nonlinear case one also finds n families of solutions, or n curves through the point $\mathbf{u}^0$. They are parametrized by $\mathbf{u}_k = \mathbf{u}_k(\xi; \mathbf{u}^0)$, with $\mathbf{u}_k(0; \mathbf{u}^0) = \mathbf{u}^0$, and we denote by $s_k = s_k(\xi; \mathbf{u}^0)$ the corresponding shock speed. To simplify notation, we will frequently write $\mathbf{u}_k(\xi)$, $s_k(\xi)$ when the point $\mathbf{u}^0$ is clearly understood. Substitution into (9.2) gives

$$\mathbf{f}\big(\mathbf{u}_k(\xi)\big) - \mathbf{f}(\mathbf{u}^0) = s_k(\xi)\big(\mathbf{u}_k(\xi) - \mathbf{u}^0\big) \ .$$

Assuming $\mathbf{u}_k$ and s_k to depend smoothly on ξ, we find after differentiating and after setting $\xi = 0$

$$D\mathbf{f}(\mathbf{u}^0)\mathbf{u}_k'(0) = s_k(0)\mathbf{u}_k'(0) \ .$$

Hence

$$\mathbf{u}_k'(0) = \alpha \mathbf{t}_k(\mathbf{u}^0) \ , \quad \alpha \in \mathbb{R} \ , \tag{9.3a}$$

$$s_k(0) = \lambda_k(\mathbf{u}^0) \ , \tag{9.3b}$$

implying that the curve $\mathbf{u}_k(\xi)$ is tangent to $\mathbf{t}_k(\mathbf{u}^0)$ at the point $\mathbf{u}^0$.

For smooth $\mathbf{f}$, we show in Chapter 10 that such solution curves exist locally in a neighbourhood of $\mathbf{u}^0$, and that the representations $\mathbf{u}_k$ and s_k are smooth. This is an application of the implicit-function theorem, see also LAX [45] or SMOLLER [67]. The solution curves $\mathbf{u}_k$ are called the Hugoniot curves and the set of all points on these curves is called the *Hugoniot locus* for the point $\mathbf{u}^0$.

If $\mathbf{u}$ lies on the k^{th} Hugoniot curve through $\mathbf{u}^0$, we say that $\mathbf{u}$ and $\mathbf{u}^0$ are connected by a k-*shock*. In many applications, with $\mathbf{f}$ explicitly given, the Hugoniot curves exist globally, that is for all relevant $\mathbf{u}$-values. This is clearly the case in the following example.

Example 9.1. *Isothermal equations of gas dynamics: i.e. equations* (7.7) *with* $\gamma = 1$. *Writing* $m = \rho v$ *for the momentum and* $a^2 = \overline{\kappa}$, *these equations become*

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial m}{\partial x} = 0 \ , \\[2ex] \dfrac{\partial m}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{m^2}{\rho} + a^2 \rho\right) = 0 \ . \end{cases} \tag{9.4}$$

Setting $\mathbf{u} = (\rho, m)^T$, *we put* (9.4) *in the form* $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0$ *and obtain for the Jacobian matrix*

$$D\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ a^2 - \dfrac{m^2}{\rho^2} & \dfrac{2m}{\rho} \end{pmatrix} \ .$$

This matrix has eigenvalues

$$\lambda_1(\mathbf{u}) = \frac{m}{\rho} - a \quad \textit{and} \quad \lambda_2(\mathbf{u}) = \frac{m}{\rho} + a \ , \tag{9.5}$$

with corresponding eigenvectors

$$\mathbf{t}_1(\mathbf{u}) = \left(1, \frac{m}{\rho} - a\right)^T \quad \textit{and} \quad \mathbf{t}_2(\mathbf{u}) = \left(1, \frac{m}{\rho} + a\right)^T \ . \tag{9.6}$$

Next we determine the Hugoniot curves $\mathbf{u}_k(\xi; \mathbf{u}^0)$ *and the speeds* $s_k(\xi; \mathbf{u}^0)$. *The Rankine–Hugoniot conditions for the states* $\mathbf{u}$ *and* $\mathbf{u}^0$ *become*

$$\begin{aligned} m - m^0 &= s(\rho - \rho^0) \ , \\ \left(\frac{m^2}{\rho} + a^2 \rho\right) - \left(\frac{m^{0^2}}{\rho^0} + a^2 \rho^0\right) &= s(m - m^0) \ . \end{aligned}$$

These equations can be solved for m *and* s *in terms of* ρ. *Writing*

$$\rho_k(\xi; \mathbf{u}^0) = \rho^0(1 + \xi) \ , \quad \textit{with} \quad \xi > -1 \quad \textit{and} \quad k = 1, 2 \ ,$$

we obtain the parametrized curves

$$\mathbf{u}_1(\xi; \mathbf{u}^0) = \mathbf{u}^0 + \xi \begin{bmatrix} \rho^0 \\ m^0 - a\rho^0\sqrt{1+\xi} \end{bmatrix} \ , \quad s_1(\xi; \mathbf{u}^0) = \frac{m^0}{\rho^0} - a\sqrt{1+\xi} \ , \tag{9.7}$$

and

$$\mathbf{u}_2(\xi; \mathbf{u}^0) = \mathbf{u}^0 + \xi \begin{bmatrix} \rho^0 \\ m^0 + a\rho^0\sqrt{1+\xi} \end{bmatrix} \ , \quad s_2(\xi; \mathbf{u}^0) = \frac{m^0}{\rho^0} + a\sqrt{1+\xi} \ , \tag{9.8}$$

where (9.7) *is the 1-curve and* (9.8) *the 2-curve. Equations* (9.7) *and* (9.8) *imply*

$$\frac{\mathrm{d}\mathbf{u}_k}{\mathrm{d}\xi}(0; \mathbf{u}^0) = \rho^0 \mathbf{t}_k(\mathbf{u}^0) \ , \quad s_k(0; \mathbf{u}^0) = \lambda_k(\mathbf{u}^0) \ , \quad k = 1, 2 \ ,$$

which is consistent with the previous results (9.3). *In Figure 9.1 some examples of Hugoniot curves are given.*

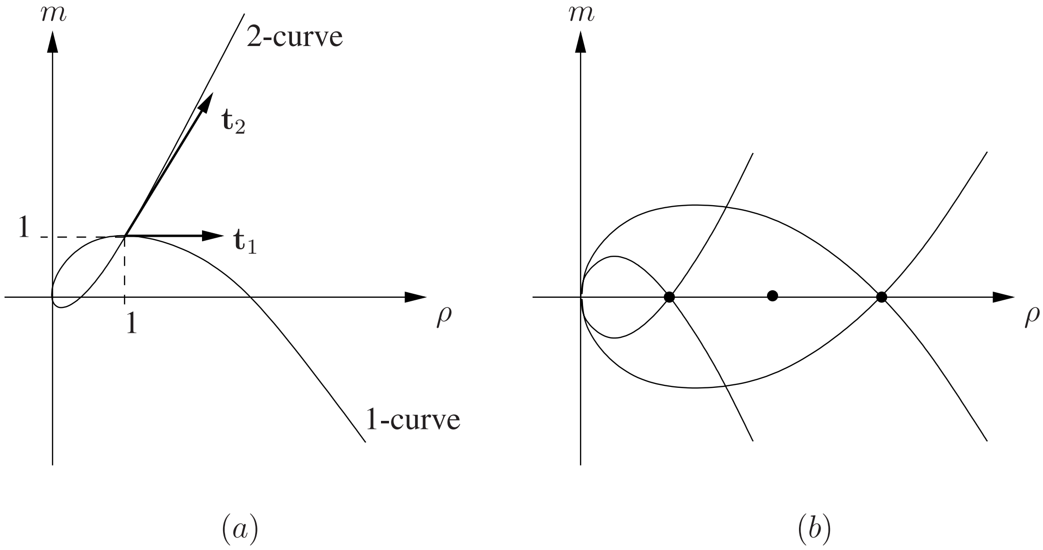


Figure 9.1. *(a) Hugoniot locus for the state $\mathbf{u}^0 = (1,1)^T$ related to the isothermal gas flow equations ($a = 1, \gamma = 1$). (b) Variation of these curves for $\mathbf{u}^0 = (\rho^0, 0)^T$ with $\rho^0 = 1, 3$*

Remark 9.2. *The curves in the Hugoniot locus for the state $\mathbf{u}^0$ are not* integral curves *(i.e. curves being tangent to the eigenvectors in all their points) of the system. This only occurs at the point $\mathbf{u}^0$. We shall encounter integral curves when dealing with rarefaction waves.*

Remark 9.3. *Every Hugoniot locus in Example 9.1 terminates at the origin, which is clearly a singular point of the system. The origin is called the* vacuum state*, since $\rho = 0$.*

Ignoring possible entropy conditions, we now give the construction of a solution of the Riemann problem involving only shocks. We first return to Example 9.1. As in the linear case, see Figure 8.6, we want to determine an intermediate state $\mathbf{u}_\mathrm{m}$ to connect $\mathbf{u}_\mathrm{l}$ to $\mathbf{u}_\mathrm{r}$. In the phase plane we draw the Hugoniot locus for $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$, see Figure 9.2, and again find two intersection points: the states $\mathbf{u}_\mathrm{m}$ and $\mathbf{u}_\mathrm{m}^*$. Thus we can connect $\mathbf{u}_\mathrm{l}$ to $\mathbf{u}_\mathrm{m}$ by a 1-shock and then to $\mathbf{u}_\mathrm{r}$ by a 2-shock or, the other path, connect $\mathbf{u}_\mathrm{l}$ to $\mathbf{u}_\mathrm{m}^*$ by a 2-shock and then to $\mathbf{u}_\mathrm{r}$ by a 1-shock. To reject one possibility we investigate the corresponding shock speeds. We find

$$s_1(\xi;\mathbf{u}_\mathrm{m}) = \frac{m_\mathrm{m}}{\rho_\mathrm{m}} - a\sqrt{1+\xi} < \frac{m_\mathrm{m}}{\rho_\mathrm{m}} \ , \quad \xi > -1 \ ,$$

and

$$s_2(\xi;\mathbf{u}_\mathrm{m}) = \frac{m_\mathrm{m}}{\rho_\mathrm{m}} + a\sqrt{1+\xi} > \frac{m_\mathrm{m}}{\rho_\mathrm{m}} \ , \quad \xi > -1 \ .$$

Knowing now that the speed of a 1-shock is smaller than the speed of a 2-shock, we reject the intersection leading to $\mathbf{u}_\mathrm{m}^*$ because the shock from $\mathbf{u}_\mathrm{l}$ to $\mathbf{u}_\mathrm{m}^*$ would travel faster than the shock going from $\mathbf{u}_\mathrm{m}^*$ to $\mathbf{u}_\mathrm{r}$: it would lead to a multi-valued solution.

To solve a Riemann problem for a general system in $\mathbb{R}^n$, one needs to determine a sequence of intermediate states $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{n-1}$ such that $\mathbf{u}_\mathrm{l}$ is connected to $\mathbf{u}_1$ by a 1-shock, $\mathbf{u}_1$ is connected

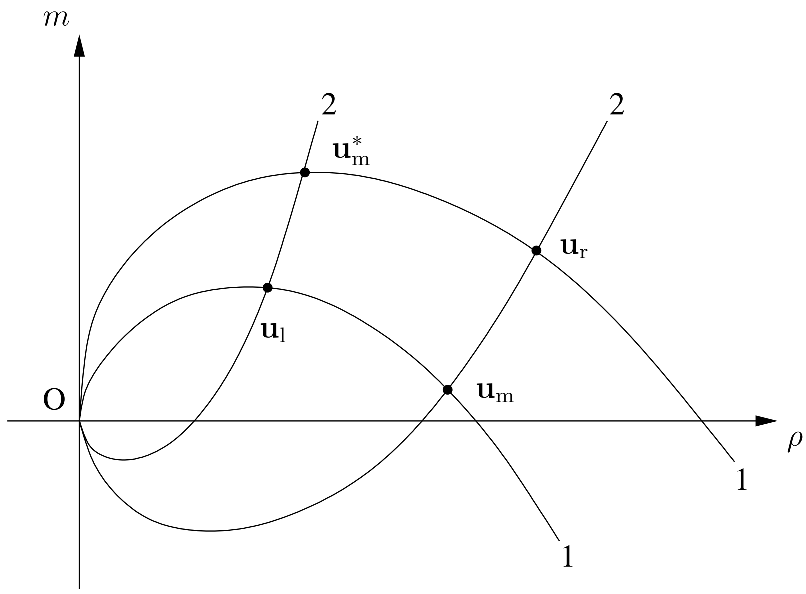


Figure 9.2. *Construction of intermediate state* $\mathbf{u}_{\mathrm{m}}$

to $\mathbf{u}_2$ by a 2-shock, $\cdots$, $\mathbf{u}_{n-1}$ is connected to $\mathbf{u}_{\mathrm{r}}$ by a n-shock. If $||\mathbf{u}_{\mathrm{l}} - \mathbf{u}_{\mathrm{r}}||$ is sufficiently small, this can always be achieved, see again Chapter 10, or [45], [67]. The idea behind it is quite simple: from any given state $\mathbf{u}^0$, we can reach a one-parameter family of states $\mathbf{u}_1(\xi_1)$ by a 1-shock. From each $\mathbf{u}_1(\xi_1)$ we can reach another one-parameter family of states $\mathbf{u}_2(\xi_1, \xi_2)$ by a 2-shock. Continuing, we find that from $\mathbf{u}^0$ we can reach a n-parameter family of states $\mathbf{u}_n(\xi_1, \cdots, \xi_n)$. Since

$$\left.\frac{\partial \mathbf{u}_n}{\partial \xi_k}\right|_{\xi_1 = \cdots = \xi_n = 0} \sim \mathbf{t}_k(\mathbf{u}^0) \ , \quad \text{for} \quad k = 1, \cdots, n \ ,$$

we know that the map $\mathbf{u}_n : \mathbb{R}^n \to \mathbb{R}^n$ is non-singular and hence bijective near the origin. Hence for any $\mathbf{u}_{\mathrm{r}}$ sufficiently close to $\mathbf{u}_{\mathrm{l}}$, there exists a unique set of parameters $\xi_1^*, \cdots, \xi_n^*$ such that $\mathbf{u}_n(\xi_1^*, \cdots, \xi_n^*) = \mathbf{u}_{\mathrm{r}}$. Note that since the eigenvalues $\lambda_k(\mathbf{u}_0)$ are ordered, the corresponding shock speeds are ordered (by continuity) as well.

Next we discuss entropy conditions.

9.2 Entropy conditions and genuine nonlinearity

In Part I we discussed various forms of the entropy condition for the scalar equation. In these discussions the convexity/concavity of f, the scalar flux function, plays a crucial role. For instance for $f'' > 0$, we pose the Lax entropy inequalities

$$f'(u_{\mathrm{r}}) < s < f'(u_{\mathrm{l}}) \ , \tag{9.9}$$

where s denotes the shockspeed and $u_{\mathrm{r}}, u_{\mathrm{l}}$ the right, left values of u at the shock. In order to generalize (9.9) to systems, we first need to generalize the convexity condition. We shall now require, see also

[45], [67],

$$\nabla\lambda_k(\mathbf{u}) \cdot \mathbf{t}_k(\mathbf{u}) \neq 0 \;, \quad \text{for all } \mathbf{u} \;, \tag{9.10}$$

and for $k = 1, \cdots, n$. If this property holds, we call the characteristic fields genuinely nonlinear. Hence the proper generalization requires not only that $\nabla\lambda_k(\mathbf{u}) \neq \mathbf{0}$, but in addition that it is not orthogonal to the corresponding eigenvector. It implies that the eigenvalues vary *monotonically* along integral curves. Without this property it is impossible to construct rarefaction waves, see Section 9.3. Conditions (9.10) are also needed to prove equivalence between vanishing viscosity solutions and entropy solutions, see [45] and Chapter 11. Before we turn to the entropy inequalities, we first consider the linear problem

$$\frac{\partial\mathbf{u}}{\partial t} + A\frac{\partial\mathbf{u}}{\partial x} = 0$$

in the quarter plane $x > 0, t > 0$. Here A is a constant $n \times n$ matrix with eigenvalues $\lambda_1 < \cdots < \lambda_k < 0 < \lambda_{k+1} < \cdots < \lambda_n$. Applying a decoupling as in Chapter 8, we find that $(n-k)$ conditions on the components of $\mathbf{u}$ must be specified along the boundary $x = 0$. More generally, if we don't have a quarter plane problem, but instead we have a boundary which moves with speed s and if $\lambda_1 < \cdots < \lambda_k < s < \lambda_{k+1} < \cdots < \lambda_n$, we must give $(n-k)$ conditions on $\mathbf{u}$ along $x = st$, to specify the solution in the region $x > st$, $t > 0$.

If we have a discontinuity of the hyperbolic system (9.1) in the x,t-plane, the above remarks can be extended. Let $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$, respectively, be the values of $\mathbf{u}$ on the left and right of a discontinuity which moves with speed s. Suppose that for some $1 \leqslant k \leqslant n$,

$$\lambda_k(\mathbf{u}_\mathrm{r}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{r}) \;.$$

To determine the solution to the right of the discontinuity, we should specify $(n-k)$ conditions on the right boundary of the shock. Similarly, looking at the left boundary, if

$$\lambda_j(\mathbf{u}_\mathrm{l}) < s < \lambda_{j+1}(\mathbf{u}_\mathrm{l}) \;,$$

for some $1 \leqslant j \leqslant n$, we must specify j conditions on the left boundary of the shock (in order to determine the solution to the left of it). Across the shock the Rankine–Hugoniot conditions (9.2) give n equations (or conditions) between $\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}$ and s. Eliminating s, leaves us with $(n-1)$ conditions between $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$. Thus we should require

$$(n-k) + j = n - 1 \qquad \text{or} \qquad j = k - 1 \;.$$

Hence we can admit a discontinuity $\{\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}, s\}$, if for some index k, $1 \leqslant k \leqslant n$,

$$\lambda_k(\mathbf{u}_\mathrm{r}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{r}) \;, \tag{9.11a}$$

$$\lambda_{k-1}(\mathbf{u}_\mathrm{l}) < s < \lambda_k(\mathbf{u}_\mathrm{l}) \;. \tag{9.11b}$$

A discontinuity $\{\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}, s\}$ across which these relations hold is called an *admissible k-shock*. We refer to inequalities (9.11) as the *Lax entropy inequalities*. Rewriting (9.11), we have:

a discontinuity $\{\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}, s\}$ is admissible (in the sense of Lax) if

- *it satisfies the Rankine–Hugoniot condition*; (9.12a)
- $\lambda_k(\mathbf{u}_\mathrm{r}) < s < \lambda_k(\mathbf{u}_\mathrm{l})$; (9.12b)
- $\lambda_{k-1}(\mathbf{u}_\mathrm{l}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{r})$, (9.12c)

for some k, $1 \leqslant k \leqslant n$.

Hence for only one index k the shock speed s is intermediate to the characteristic speeds λ_k on both sides of the shock.

What are the consequences of (9.12) for the shock solution in Example 9.1. In other words, which of the intermediate states $\mathbf{u}_{\mathrm{m}}$ as shown in Figure 9.2 give admissible shocks? Along 1-curves (giving 1-shocks) we impose

$$\lambda_1(\mathbf{u}_{\mathrm{r}}) < s < \lambda_1(\mathbf{u}_{\mathrm{l}}) \; , \tag{9.13a}$$

$$s < \lambda_2(\mathbf{u}_{\mathrm{r}}) \; , \tag{9.13b}$$

and along 2-curves (giving 2-shocks) we impose

$$\lambda_2(\mathbf{u}_{\mathrm{r}}) < s < \lambda_2(\mathbf{u}_{\mathrm{l}}) \; , \tag{9.13c}$$

$$\lambda_1(\mathbf{u}_{\mathrm{l}}) < s \; . \tag{9.13d}$$

Before we investigate the implications of (9.13) we first need to verify the genuine nonlinearity (9.10) for $k=1,2$. This is satisfied, since

$$\nabla\lambda_1(\mathbf{u}) \cdot \mathbf{t}_1(\mathbf{u}) = -\frac{a}{\rho} < 0 \; ,$$
$$\nabla\lambda_2(\mathbf{u}) \cdot \mathbf{t}_2(\mathbf{u}) = +\frac{a}{\rho} > 0 \; .$$

Using (9.5) and (9.7) in (9.13a-9.13b) results in the inequalities

$$\frac{m_{\mathrm{r}}}{\rho_{\mathrm{r}}} - a < \frac{m_{\mathrm{r}}}{\rho_{\mathrm{r}}} - a\sqrt{\frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{r}}}} = \frac{m_{\mathrm{l}}}{\rho_{\mathrm{l}}} - a\sqrt{\frac{\rho_{\mathrm{r}}}{\rho_{\mathrm{l}}}} < \frac{m_{\mathrm{l}}}{\rho_{\mathrm{l}}} - a \tag{9.14}$$

and

$$\frac{m_{\mathrm{r}}}{\rho_{\mathrm{r}}} - a\sqrt{\frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{r}}}} < \frac{m_{\mathrm{r}}}{\rho_{\mathrm{r}}} + a \; . \tag{9.15}$$

The inequalities in (9.14) are satisfied if

$$\rho_{\mathrm{l}} < \rho_{\mathrm{r}} \qquad \text{(for 1-shocks)} \; ,$$

and (9.15) imposes no restriction. Similarly we obtain admissible shocks along a 2-curve only if

$$\rho_{\mathrm{l}} > \rho_{\mathrm{r}} \qquad \text{(for 2-shocks)} \; .$$

Remark 9.4. *The velocity of the fluid is given by $v = m/\rho$. Hence the expressions for the shock speed imply for 1-shocks*

$$s = \frac{m_{\mathrm{r}}}{\rho_{\mathrm{r}}} - a\sqrt{\frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{r}}}} < v_{\mathrm{r}}$$

and also

$$s = \frac{m_\mathrm{l}}{\rho_\mathrm{l}} - a\sqrt{\frac{\rho_\mathrm{r}}{\rho_\mathrm{l}}} < v_\mathrm{l} \,.$$

Thus the fluid velocity on both sides of the shock exceeds the shock speed: a fluid particle moves through the shock from left to right. Thus the entropy conditions require that the density increases when passing through a shock. For 2-shocks we have

$$s = \frac{m_\mathrm{r}}{\rho_\mathrm{r}} + a\sqrt{\frac{\rho_\mathrm{l}}{\rho_\mathrm{r}}} > v_\mathrm{r}$$

and also

$$s = \frac{m_\mathrm{l}}{\rho_\mathrm{l}} + a\sqrt{\frac{\rho_\mathrm{r}}{\rho_\mathrm{l}}} > v_\mathrm{l} \,.$$

Hence 2-shocks travel faster than the fluid velocity on either side: now a fluid particle moves through the shock from right to left. Consequently, the entropy conditions (implying $\rho_\mathrm{l} > \rho_\mathrm{r}$) result again in a situation where the density increases when fluid passes through a shock.

In Figure 9.3 we show the admissible shocks for given states $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$. Note that here the interpretation of left and right is of crucial importance.

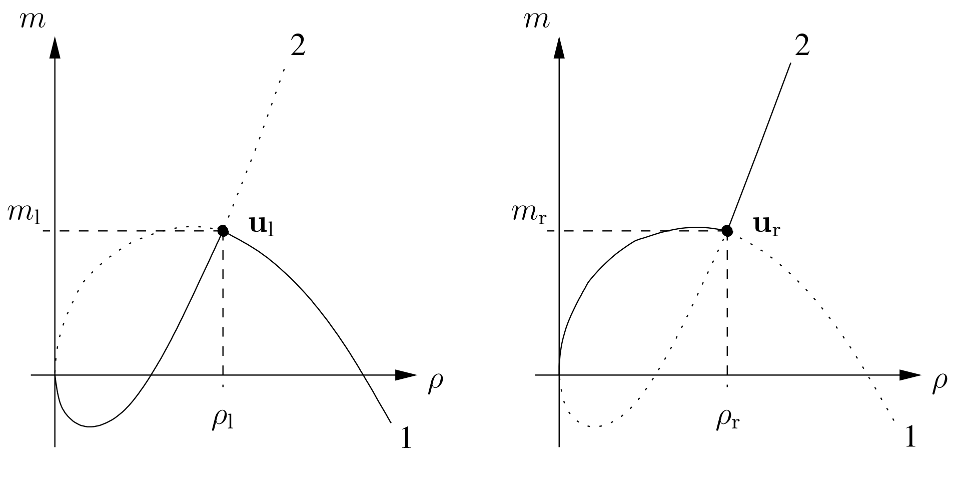


Figure 9.3. *Admissible shock-curves for given left and right states*

With reference to Figure 9.2, we now see that only the shock from $\mathbf{u}_\mathrm{l}$ to $\mathbf{u}_\mathrm{m}$ is admissible. The "non-physical" shock from $\mathbf{u}_\mathrm{m}$ to $\mathbf{u}_\mathrm{r}$ needs to be replaced by a rarefaction wave. Combining both admissible possibilities in Figure 9.3, we can construct for a given state $\mathbf{u}_\mathrm{l}$, the $\mathbf{u}_\mathrm{r}$-region for which the solution of the Riemann problem consists of two entropy shocks. This is shown in Figure 9.4.

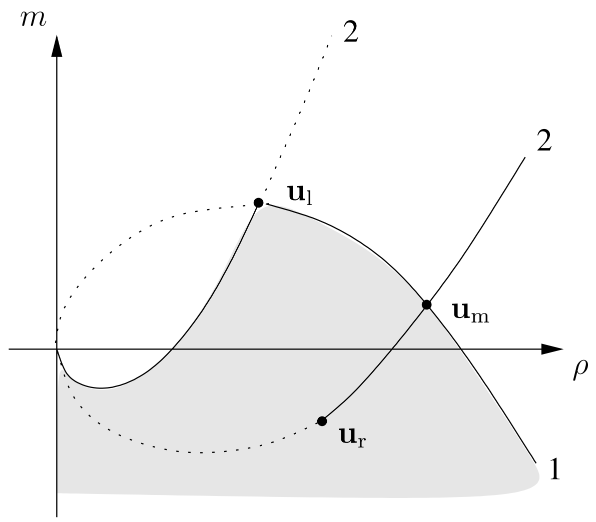


Figure 9.4. *Taking $\mathbf{u}_r$ from the shaded region results in a solution of the Riemann problem with two admissible shocks*

Remark 9.5. *For a constant coefficient, linear system the condition of genuine nonlinearity is violated for all k because the eigenvalues are constant. More generally, for a nonlinear system it might happen that in one of the characteristic fields $\mathbf{t}_k(\mathbf{u})$ the eigenvalue $\lambda_k(\mathbf{u})$ is constant along integral curves of that field. Hence*

$$\nabla\lambda_k(\mathbf{u}) \cdot \mathbf{t}_k(\mathbf{u}) = 0 \,, \quad \forall\, \mathbf{u} \,.$$

Then we say that the k^{th}-field is linearly degenerate*. A discontinuity in a linear degenerate field is called a* contact-discontinuity*.*

9.3 Rarefaction waves

It is clear from the discussion in the previous section that in general the solution of a Riemann problem cannot consist of shocks only. In order to develop a full solution satisfying the entropy conditions, one has to incorporate rarefaction waves as well.

Generalizing the scalar case, we call a solution of (R) a rarefaction wave if

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_l & \text{for } x \leqslant \eta_l t \,, \\ \mathbf{w}(x/t) & \text{for } \eta_l t < x < \eta_r t \,, \\ \mathbf{u}_r & \text{for } x \geqslant \eta_r t \,, \end{cases} \tag{9.16}$$

where $-\infty \leqslant \eta_l < \eta_r \leqslant \infty$ and $\mathbf{w} : (\eta_l, \eta_r) \to \mathbb{R}^n$ is a smooth vector valued function satisfying $\lim\limits_{\eta\downarrow\eta_l} \mathbf{w}(\eta) = \mathbf{u}_l$ and $\lim\limits_{\eta\uparrow\eta_r} \mathbf{w}(\eta) = \mathbf{u}_r$. Substituting (9.16) into equation (9.1) yields the expression

$$D\mathbf{f}\big(\mathbf{w}(\eta)\big)\,\mathbf{w}'(\eta) = \eta\,\mathbf{w}'(\eta)$$

for $\eta_{\mathrm{l}}<\eta<\eta_{\mathrm{r}}$. This implies for some k, $1\leqslant k\leqslant n$:

$$(i)\quad \mathbf{w}'(\eta)=\alpha(\eta)\mathbf{t}_k\big(\mathbf{w}(\eta)\big)\ , \tag{9.17a}$$
$$(ii)\quad \lambda_k\big(\mathbf{w}(\eta)\big)=\eta\ . \tag{9.17b}$$

Hence the values of $\mathbf{w}(\eta)$ all lie along an integral curve of $\mathbf{t}_k$. In particular the states $\mathbf{u}_{\mathrm{l}}=\mathbf{w}(\eta_{\mathrm{l}}+)$ and $\mathbf{u}_{\mathrm{r}}=\mathbf{w}(\eta_{\mathrm{r}}-)$ have to lie on the same integral curve. This is a necessary condition! However it is not sufficient. Along the integral curve η increases monotonically from η_{l} to η_{r}. Thus $\lambda_k\big(\mathbf{w}(\cdot)\big)$ has to increase along that curve as well. From a given state $\mathbf{u}_{\mathrm{l}}$ we can move along the integral curve only in the direction in which λ_k increases. If λ_k has a local maximum at $\mathbf{u}_{\mathrm{l}}$ in the direction $\mathbf{t}_k(\mathbf{u}_{\mathrm{l}})$, there are no rarefaction waves with left state $\mathbf{u}_{\mathrm{l}}$. In the generic nonlinear case, there is a one-parameter family of states that can be connected to $\mathbf{u}_{\mathrm{l}}$ by a k-rarefaction wave, all those states lying on the integral curve of $\mathbf{t}_k$ in the direction of increasing λ_k up to the next local maximum of λ_k.

If the k^{th}-field is genuinely nonlinear, then λ_k is monotonically varying along the entire integral curve. In that case we see that $\mathbf{u}_{\mathrm{l}}$ and $\mathbf{u}_{\mathrm{r}}$ can always be connected by a rarefaction wave provided they belong to the same integral curve and $\lambda_k(\mathbf{u}_{\mathrm{l}})<\lambda_k(\mathbf{u}_{\mathrm{r}})$. If the k^{th}-field is linearly degenerate, then λ_k is constant along the integral curve and no rarefaction waves are possible in this family. If all fields are genuinely nonlinear, there exist n families of states that can be connected to $\mathbf{u}_{\mathrm{l}}$ by a rarefaction wave.

The vector $\mathbf{w}(\eta)$ is computed as follows. Differentiating (9.17b) with respect to η yields for $\eta_{\mathrm{l}}<\eta<\eta_{\mathrm{r}}$

$$\nabla\lambda_k\big(\mathbf{w}(\eta)\big)\cdot\mathbf{w}'(\eta)=1\ ,$$

or

$$\nabla\lambda_k\big(\mathbf{w}(\eta)\big)\cdot\mathbf{t}_k\big(\mathbf{w}(\eta)\big)\alpha(\eta)=1\ .$$

Hence

$$\alpha(\eta)=\frac{1}{\nabla\lambda_k\big(\mathbf{w}(\eta)\big)\cdot\mathbf{t}_k\big(\mathbf{w}(\eta)\big)}\ .$$

Substituting this expression in (9.17a) gives the ordinary differential equations

$$\mathbf{w}'=\frac{\mathbf{t}_k(\mathbf{w})}{\nabla\lambda_k(\mathbf{w})\cdot\mathbf{t}_k(\mathbf{w})}\qquad\text{for } \eta_{\mathrm{l}}<\eta<\eta_{\mathrm{r}}\ , \tag{9.18}$$

subject to $\mathbf{w}(\eta_{\mathrm{l}})=\mathbf{u}_{\mathrm{l}}$. The constants $\eta_{\mathrm{l}},\eta_{\mathrm{r}}$ follow from (9.17b): $\eta_{\mathrm{l}}=\lambda_k(\mathbf{u}_{\mathrm{l}})$ and $\eta_{\mathrm{r}}=\lambda_k(\mathbf{u}_{\mathrm{r}})$.

Next we return to the equations in Example 9.1 and compute the rarefaction waves. Using (9.5), (9.6) and setting $\mathbf{w}(\eta)=\big(\rho(\eta),m(\eta)\big)^T$, we find for the 1-rarefactions the differential equations (putting $k=1$ in (9.18))

$$\rho'(\eta)=-\rho(\eta)/a\ ,$$
$$m'(\eta)=-m(\eta)/a+\rho(\eta)\ .$$

For the moment we solve these equations for $\eta \in \mathbb{R}$, subject to $\big(\rho(\eta_1), m(\eta_1)\big) = (\rho_l, m_l)$ where $\eta_1 = \lambda_1(\mathbf{u}_l) = \dfrac{m_l}{\rho_l} - a$. This gives

$$\rho(\eta) = \rho_l \exp\{-(\eta - \eta_1)/a\} \tag{9.19}$$

and

$$\begin{aligned} m(\eta) &= \{\rho_l(\eta - \eta_1) + m_l\} \exp\{-(\eta - \eta_1)/a\} \\ &= \rho_l(\eta + a) \exp\{-(\eta - \eta_1)/a\} \ . \end{aligned} \tag{9.20}$$

As a direct consequence, the 1-integral curve has the parametrization

$$\rho > 0\ , \quad m = m(\rho) = \frac{m_l}{\rho_l}\rho - a\rho \log \frac{\rho}{\rho_l}\ . \tag{9.21}$$

This implies

$$\lambda_1(\mathbf{w}) = \lambda_1\big(\rho, m(\rho)\big) = \eta_1 - a \log \frac{\rho}{\rho_l}\ ,$$

which shows that λ_1 decreases along the 1-integral curve. Thus only that part of the curve where $\rho < \rho_l$ can be used for the purpose of constructing a 1-rarefaction.

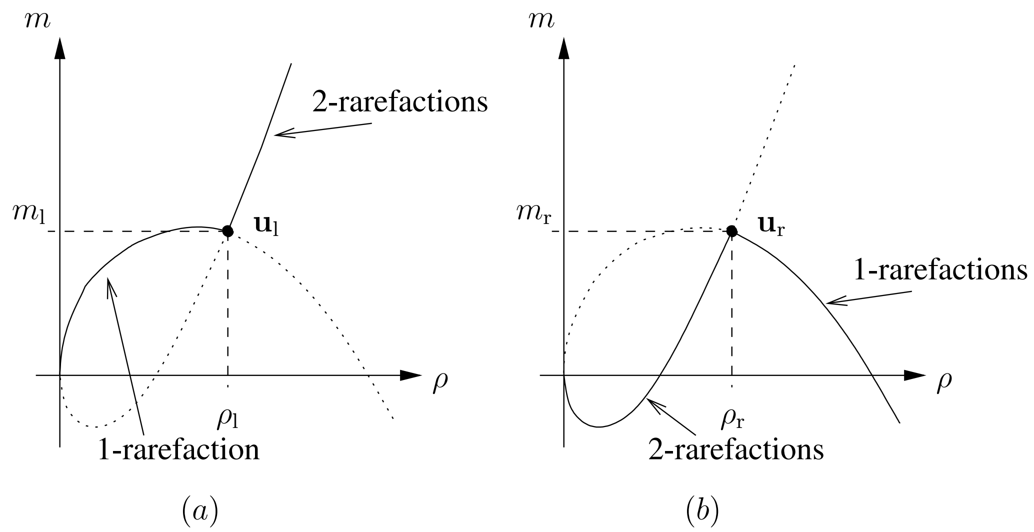


Figure 9.5. *(a) Set of states (solid lines) that can be connected to $\mathbf{u}_l$ by a rarefaction wave; (b) Set of states (solid lines) that can be connected to $\mathbf{u}_r$ by a rarefaction wave*

For 2-rarefactions we find

$$\begin{aligned} \rho(\eta) &= \rho_l \exp\{(\eta - \eta_1)/a\}\ , \\ m(\eta) &= \rho_l(\eta - a) \exp\{(\eta - \eta_1)/a\}\ , \end{aligned}$$

and consequently for the 2-integral curve

$$m(\rho) = \frac{m_l}{\rho_l}\rho + a\rho \log \frac{\rho}{\rho_l}\ .$$

This implies

$$\lambda_2(\mathbf{w}) = \lambda_2\big(\rho, m(\rho)\big) = \eta_1 + a \log \frac{\rho}{\rho_\mathrm{l}} \ .$$

Hence λ_2 increases along the 2-integral curve and only that part where $\rho > \rho_\mathrm{l}$ can be used to obtain a 2-rarefaction wave. These results are summarized in Figure 9.5. Note that integral curves are different from the shock curves in a Hugoniot-locus: only at the point $\mathbf{u}_\mathrm{l}$ (or $\mathbf{u}_\mathrm{r}$) their tangents coincide. For a given point $\mathbf{u}^*$, we can now determine the set of all points that can be connected to $\mathbf{u}^*$ by a shock or a rarefaction. This is done by putting together the curves from Figures 9.3 and 9.5. Interpreting $\mathbf{u}^*$ as $\mathbf{u}_\mathrm{l}$ gives Figure 9.6(a), $\mathbf{u}^*$ as $\mathbf{u}_\mathrm{r}$ gives Figure 9.6(b). In these figures the curves S_p denote the p-shocks and R_p the p-rarefactions: at the points $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ they match up smoothly. The solution of the general

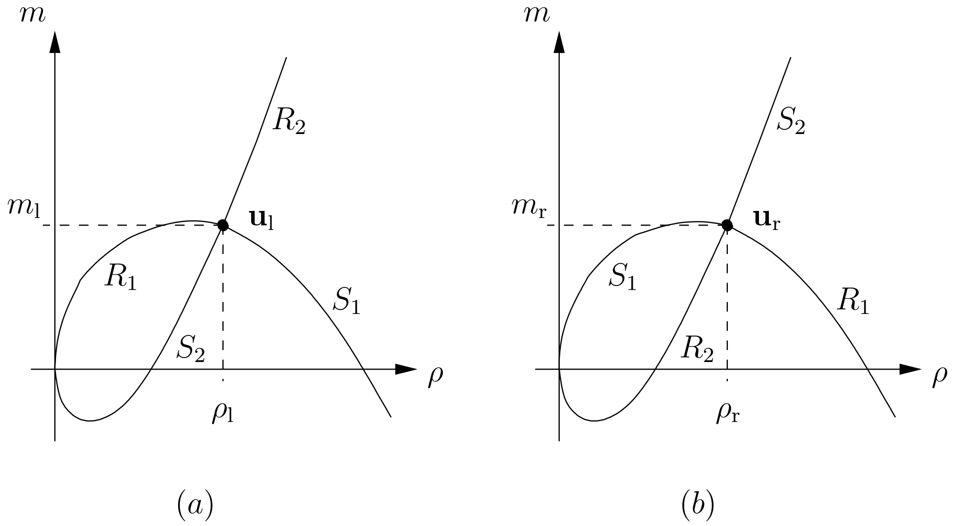


Figure 9.6. *(a) Set of admissible states for* $\mathbf{u}_\mathrm{l}$; *(b) Set of admissible states for* $\mathbf{u}_\mathrm{r}$

Riemann problem for the isothermal equations of gas dynamics (9.4) is found by combining Figures 9.6(a) and (b). This is done in Figure 9.7 for an arbitrary pair of states $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$. Again there are two intersection points, giving the possible intermediate states $\mathbf{u}_\mathrm{m}$ and $\mathbf{u}_\mathrm{m}^*$. However, as a general rule, we always have first a 1-wave followed by a 2-wave (shock or rarefaction). The other possibility leads to multivaluedness in the solution and is therefore rejected. Indeed, for the state $\mathbf{u}_\mathrm{m}$ we have

$$S_1(\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{m}) = \frac{m_\mathrm{m}}{\rho_\mathrm{m}} - a\sqrt{\frac{\rho_\mathrm{l}}{\rho_\mathrm{m}}} < \frac{m_\mathrm{m}}{\rho_\mathrm{m}} + a = \lambda_2(\mathbf{u}_\mathrm{m}) = \eta_\mathrm{m} < \lambda_2(\mathbf{u}_\mathrm{r}) = \eta_\mathrm{r} \ .$$

The solution in the $x - t$ plane is shown in Figure 9.8.

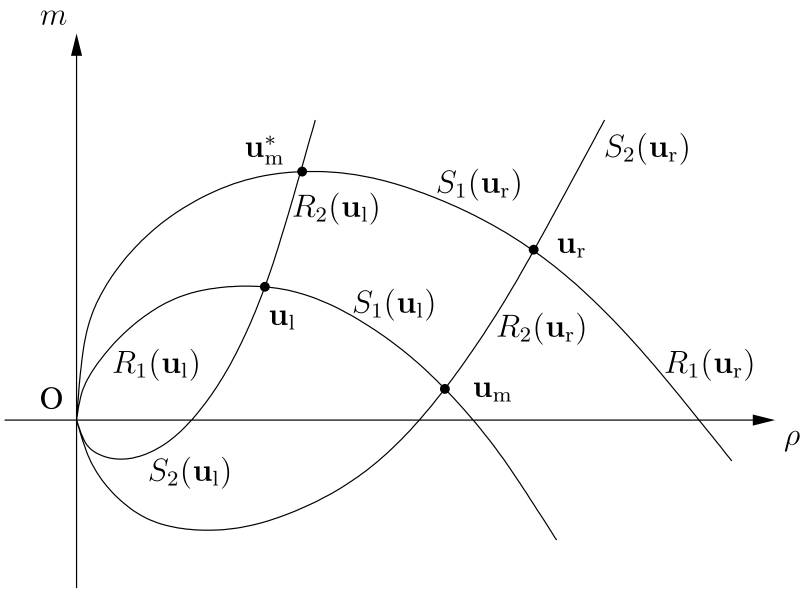


Figure 9.7. *Construction of the admissible entropy solution of the Riemann problem for the isothermal equations of gas dynamics*

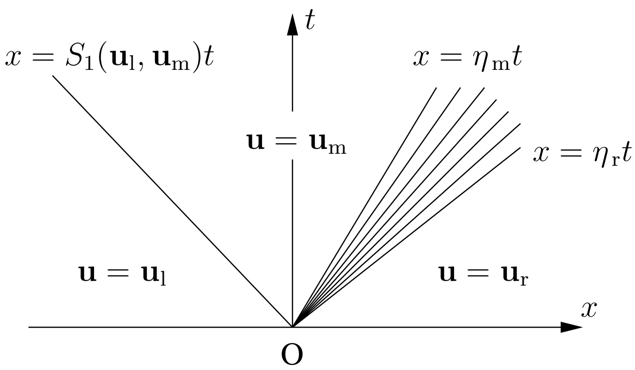


Figure 9.8. *Solution of Riemann problem in $x-t$ plane*

10 Riemann problem for nonlinear equations: the theory

In the previous chapter we discussed in detail the construction of the entropy solution. Following SMOLLER [67] and LAX [45], we now consider the existence and uniqueness questions.
Throughout we assume that the flux vector function

$$\mathbf{f} : N \subseteq \mathbb{R}^n \to \mathbb{R}^n \;, \qquad \text{for some neighbourhood } N \;,$$

is smooth, e.g. $\mathbf{f} \in C^2(N)$, and satisfies the hyperbolicity conditions. Thus the Jacobian matrix $D\mathbf{f}(\mathbf{u})$ has real, distinct eigenvalues $\lambda_1(\mathbf{u}) < \cdots < \lambda_n(\mathbf{u})$ and corresponding to each $\lambda_k(\mathbf{u})$ we have a right (column) eigenvector $\mathbf{t}_k(\mathbf{u})$ and a left (row) eigenvector $\big(\boldsymbol{\ell}_k(\mathbf{u})\big)^T$. All the eigenvalues and eigenvectors are smooth functions of $\mathbf{u}$. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we use the inner-product notation

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = \mathbf{a}^T \mathbf{b} \;.$$

We observe that

$$\boldsymbol{\ell}_i(\mathbf{u}) \cdot \mathbf{t}_j(\mathbf{u}) = 0 \quad \text{for all } 1 \leqslant i, j \leqslant n \;, \quad i \neq j \;, \tag{10.1}$$

and

$$\boldsymbol{\ell}_k(\mathbf{u}) \cdot \mathbf{t}_k(\mathbf{u}) \neq 0 \quad \text{for all } 1 \leqslant k \leqslant n \;. \tag{10.2}$$

These facts follow from some simple linear algebra. If

$$A\mathbf{x} = \lambda \mathbf{x} \quad \text{and} \quad \mathbf{y}^T A = \mu \mathbf{y}^T \quad \text{with } \lambda \neq \mu \neq 0 \;,$$

we deduce from

$$\mathbf{y}^T \mathbf{x} = \frac{1}{\mu} \mathbf{y}^T A \mathbf{x} = \frac{\lambda}{\mu} \mathbf{y}^T \mathbf{x}$$

that $\mathbf{y}^T \mathbf{x} = 0$, implying (10.1). Since all $\mathbf{t}_k$ are linearly independent, (10.2) is immediate.

An important role in the general theory is played by Riemann invariants. In Section 10.1 we give the definition, an existence result and some examples. Then we consider rarefaction waves in Section 10.2 and shock waves in Section 10.3. The solvability of the Riemann problem, for $||\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l}||$ small, is discussed in Section 10.4.

10.1 Riemann invariants

We begin with

Definition 10.1. *Let $1 \leqslant k \leqslant n$. A k-Riemann invariant is a smooth function $w : N \to \mathbb{R}$, satisfying*

$$\mathbf{t}_k \cdot \nabla w = 0 \quad \textit{in } N \ ,$$

i.e. w is constant along the k-integral curve.

The following proposition says that there are precisely $n-1$ Riemann invariants along each integral curve.

Proposition 10.2. *There exist $(n-1)$ k-Riemann invariants whose gradients are linearly independent in N.*

Proof. For any smooth function $w : N \to \mathbb{R}$ (with N sufficiently small) and $1 \leqslant k \leqslant n$, we consider

$$T_k w := \mathbf{t}_k \cdot \nabla w \quad \text{in } N \ .$$

In N we can choose a coordinate system $\{Z_1, \cdots, Z_n\}$, with $Z_i = Z_i(\mathbf{u})$ for $i = 1, \cdots, n$, such that $T_k w = \dfrac{\partial w}{\partial Z_1}$.

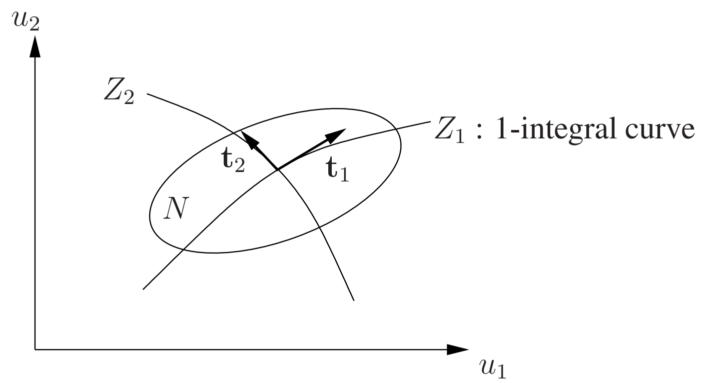


Figure 10.1. *Definition of coordinate system $\{Z_1, Z_2\}$ for $k = 1$*

Then define in N the functions $w_i = Z_{i+1}$, for $i = 1, \cdots, n-1$. This gives

$$T_k w_i = \frac{\partial w_i}{\partial Z_1} = \frac{\partial Z_{i+1}}{\partial Z_1} = 0$$

and

$$\nabla w_i = \left(\frac{\partial w_i}{\partial Z_1}, \cdots, \frac{\partial w_i}{\partial Z_n} \right) = (0, \cdots, 1, \cdots, 0) \ ,$$

for $1 \leqslant i \leqslant n-1$, showing that the gradients are linearly independent. $\square$

Below we give two examples to clarify the construction

Example 10.3. *Riemann invariants for the isothermal gas flow equations. Using* (9.21) *we have* $(\rho, m) \in$ *1-integral curve if and only if* $\rho > 0$ *and* $m(\rho) = \frac{m_l}{\rho_l}\rho - a\rho\log\frac{\rho}{\rho_l}$*. Hence*

$$\frac{m}{\rho} + a\log\rho = \frac{m_l}{\rho_l} + a\log\rho_l = \textit{constant}$$

along a 1-integral curve, implying that

$$w_1(\rho, m) = \frac{m}{\rho} + a\log\rho$$

is the 1-Riemann invariant. Similarly we obtain that

$$w_2(\rho, m) = \frac{m}{\rho} - a\log\rho$$

is the 2-Riemann invariant. Indeed, using (9.6) *one easily verifies*

$$\mathbf{t}_1 \cdot \nabla w_1 = \mathbf{t}_2 \cdot \nabla w_2 = 0 \quad \textit{in } \{\rho > 0\} \; .$$

Example 10.4. *Riemann invariants for the equations of motion of an ideal fluid. Recalling the equations, we have*

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \qquad &= 0 \; , \\ \frac{\partial\rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 + p) &= 0 \; , \\ \frac{\partial\rho S}{\partial t} + \frac{\partial}{\partial x}(\rho v S) \quad\;\; &= 0 \; , \end{aligned}$$

where $p = p(S, \rho)$ *with* $\frac{\partial p}{\partial\rho} > 0$*. Introducing the sound speed* $c := \sqrt{\frac{\partial p}{\partial\rho}}$*, we find that the Jacobian matrix*

$$\begin{pmatrix} v & \rho & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial\rho} & v & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & v \end{pmatrix}$$

has eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1 &= v - c \quad \textit{with} \quad \mathbf{t}_1 = (\rho, -c, 0)^T \; , \\ \lambda_2 &= v \qquad\;\; \textit{with} \quad \mathbf{t}_2 = \left(\frac{\partial p}{\partial S}, 0, -\frac{\partial p}{\partial\rho}\right)^T \; , \\ \lambda_3 &= v + c \quad \textit{with} \quad \mathbf{t}_3 = (\rho, c, 0)^T \; . \end{aligned}$$

Here $n = 3$*, thus we are looking for* $3 \times 2 = 6$ *Riemann invariants. They are given by*

$$\{S, v + h\} \; , \quad \{v, p\} \; , \quad \{S, v - h\} \; ,$$

where $h = h(\rho, S)$ is the enthalpy, which satisfies $\dfrac{\partial h}{\partial \rho} = \dfrac{c}{\rho}$. For example, the Riemann invariants along the 3-integral curve must satisfy

$$\mathbf{t}_3 \cdot \nabla w = \rho \frac{\partial w}{\partial \rho} + c \frac{\partial w}{\partial v} = 0 \ .$$

Clearly $w = S$ (as an independent variable) satisfies this equation, as does $w = v - h$:

$$\rho \frac{\partial w}{\partial \rho} + c \frac{\partial w}{\partial v} = -\rho \frac{\partial h}{\partial \rho} + c = 0 \ .$$

10.2 Rarefaction waves

Consider the following definition.

Definition 10.5. *Fix $1 \leqslant k \leqslant n$. Let $\mathbf{u}$ be a C^1 solution of equation* (9.1) *in a domain $D \subseteq Q$ and suppose that all k-Riemann invariants are constant in D, i.e. $w_i\big(\mathbf{u}(x,t)\big) = c_i$ for $i = 1, \cdots, n-1$ and for all $(x,t) \in D$. Then $\mathbf{u}$ is called a k-rarefaction (wave).*

In this definition a smooth solution $\mathbf{u}$ is a k-rarefaction if the values $\mathbf{u}(x,t)$ belong to the intersection of the $(n-1)$ surfaces $w_i(\mathbf{u}) = c_i$. Because the gradients ∇w_i are linearly independent, the $n-1$ equations

$$w_i(u_1, \cdots, u_n) = c_i \ , \qquad 1 \leqslant i \leqslant n-1 \ ,$$

in the n unknowns define a curve in the set N, see Figure 10.2

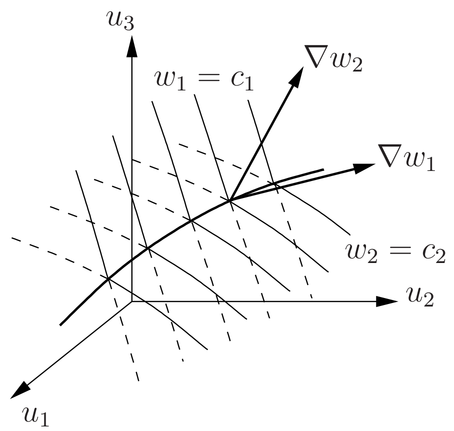


Figure 10.2. *Intersection of Riemann invariants w_1 and w_2*

The intersection curve must be the k-integral curve. Indeed, if $\mathbf{u}(\theta)$ denotes a local parametrization, then

$$\frac{\mathrm{d}}{\mathrm{d}\theta} w_i\big(\mathbf{u}(\theta)\big) = \nabla w_i \cdot \mathbf{u}'(\theta) = 0 \ , \quad \text{for all } 1 \leqslant i \leqslant n-1 \ .$$

Hence $\mathbf{u}'(\theta)$ must be orthogonal to the $n-1$ dimensional space spanned by the $\nabla w_i\big(\mathbf{u}(\theta)\big)$. Consequently $\mathbf{u}'(\theta) \in \operatorname{span}\{\mathbf{t}_k\big(\mathbf{u}(\theta)\big)\}$ which proves the assertion. In the following proposition, the operation

$$\frac{\mathrm{d}}{\mathrm{d}k} := \frac{\partial}{\partial t} + \lambda_k \frac{\partial}{\partial x}$$

denotes differentiation in the k-characteristic direction.

Proposition 10.6. *A function* $\mathbf{u}$ *is a* C^1 *solution of* (9.1) *in* $D \subseteq Q$ *if and only if* $\boldsymbol{\ell}_k^T \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}k} = 0$ *in* D *for all* $1 \leqslant k \leqslant n$.

Proof. Clearly, $\mathbf{u}$ is a C^1 solution of (9.1) in D if and only if

$$\frac{\partial \mathbf{u}}{\partial t} + D\mathbf{f}(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{in } D\ .$$

Multiplying the equation on the left by $\boldsymbol{\ell}_k^T$ yields

$$\boldsymbol{\ell}_k^T \frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\ell}_k^T D\mathbf{f}(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = \boldsymbol{\ell}_k^T \left\{ \frac{\partial \mathbf{u}}{\partial t} + \lambda_k \frac{\partial \mathbf{u}}{\partial x} \right\} = \boldsymbol{\ell}_k^T \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}k}\ . \tag{10.3}$$

We now conclude the assertion: if $\mathbf{u}$ solves (9.1) in D, then (10.3) must be zero in D for all $1 \leqslant k \leqslant n$. On the other hand, if (10.3) is zero in D for $1 \leqslant k \leqslant n$, then

$$\frac{\partial \mathbf{u}}{\partial t} + D\mathbf{f}(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} \perp \operatorname{span}\{\boldsymbol{\ell}_k : 1 \leqslant k \leqslant n\} = \mathbb{R}^n\ ,$$

which shows that $\mathbf{u}$ is a classical solution in D. □

Rarefaction waves have the following characteristic property.

Proposition 10.7. *Let* $\mathbf{u}$ *be a k-rarefaction in* $D \subseteq Q$. *Then the k-characteristics (i.e. curves satisfying* $\dot{x} = \lambda_k\big(\mathbf{u}(x,t)\big)$*) are straight lines along which* $\mathbf{u}$ *is constant.*

Proof. By the previous result we know that

$$\boldsymbol{\ell}_k^T \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}k} = 0 \quad \text{in } D\ .$$

By definition, the $n-1$ Riemann invariants $\{w_1, \cdots, w_{n-1}\}$ are constant in D. This implies

$$0 = \frac{\mathrm{d}w_i}{\mathrm{d}k} = \nabla w_i \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}k} = 0 \quad \text{in } D\ ,$$

for $1 \leqslant i \leqslant n-1$. Since the ∇w_i are linearly independent, $\nabla w_i \cdot \mathbf{t}_k = 0$ for all $1 \leqslant i \leqslant n-1$ and $\boldsymbol{\ell}_k \cdot \mathbf{t}_k \neq 0$, we see that

$$\operatorname{span}\{\nabla w_1, \cdots, \nabla w_{n-1}, \boldsymbol{\ell}_k\} = \mathbb{R}^n\ ,$$

which implies

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}k} = 0 \quad \text{in } D\ .$$

Thus $\mathbf{u}$ is constant in the k-characteristic direction and the k-characteristic curves are straight lines. □

Knowing that a k-rarefaction in D satisfies

$$\begin{cases} \mathbf{u}\big(x(t),t\big) = \mathbf{c} \quad \text{(constant)} , \\ \dot{x}(t) = \lambda_k(\mathbf{c}) , \end{cases}$$

for $\big(x(t),t\big) \in D$, we now consider a particular class, called centered k-rarefactions.

Definition 10.8. *A centered k-rarefaction, centered at $(x_0,t_0) \in \overline{Q}$, is a k-rarefaction depending only on $(x-x_0)/(t-t_0)$.*

We present a first existence result, related to centered k-rarefactions. For this we need that the k^{th}-characteristic field is genuinely nonlinear, see (9.10). Throughout we assume that $\mathbf{t}_k$ is normalized so that

$$\nabla\lambda_k(\mathbf{u}) \cdot \mathbf{t}_k(\mathbf{u}) = 1 \quad \text{for } \mathbf{u} \in N \ . \tag{10.4}$$

We have

Theorem 10.9. *Let* (10.4) *hold and let $\mathbf{u}_\mathrm{l}$ be any point in N. Then there exists a one-parameter family of states $\mathbf{u}=\mathbf{u}(\varepsilon)$, with $0<\varepsilon<a$ (small) and $\mathbf{u}(0)=\mathbf{u}_\mathrm{l}$, which can be connected on the right to $\mathbf{u}_\mathrm{l}$ by a centered k-rarefaction.*

Proof. Since all the $(n-1)$ k-Riemann invariants are constant for the values of the k-rarefaction, we must look for

$$w_i(\mathbf{u}) = w_i(\mathbf{u}_\mathrm{l}) \quad \text{for } i = 1, \cdots, n-1 \ .$$

We introduce a parameter ε by setting

$$\lambda_k(\mathbf{u}) = \lambda_k(\mathbf{u}_\mathrm{l}) + \varepsilon \ ,$$

and we consider the map $\mathbf{F}: N \times \mathbb{R} \to \mathbb{R}^n$ given by

$$\mathbf{F}(\mathbf{u},\varepsilon) = \big(w_1(\mathbf{u}) - w_1(\mathbf{u}_\mathrm{l}), \cdots, w_{n-1}(\mathbf{u}) - w_{n-1}(\mathbf{u}_\mathrm{l}), \lambda_k(\mathbf{u}) - \lambda_k(\mathbf{u}_\mathrm{l}) - \varepsilon\big) \ .$$

For $\mathbf{u} \in N$, $\varepsilon \in \mathbb{R}$ we want to solve

$$\mathbf{F}(\mathbf{u},\varepsilon) = 0 \ .$$

Since the n columns

$$\big[\nabla w_1, \cdots, \nabla w_{n-1}, \nabla\lambda_k\big]$$

in the Jacobian matrix are linearly independent, this is possible for $|\varepsilon|$ small, say $|\varepsilon| < a$. This is a consequence of the implicit function theorem (e.g. see RUDIN [64]). It gives a differentiable curve $\mathbf{u}=\mathbf{u}(\varepsilon;\mathbf{u}_\mathrm{l})$, satisfying $\mathbf{u}(0)=\mathbf{u}_\mathrm{l}$.
Any k-Riemann invariant w is constant along this curve: hence

$$\frac{\mathrm{d}w}{\mathrm{d}\varepsilon} = \nabla w \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} = 0 \ .$$

Thus

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} \perp \operatorname{span}\{\nabla w_1, \cdots, \nabla w_{n-1}\}$$

implying

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} = \alpha(\varepsilon)\mathbf{t}_k\big(\mathbf{u}(\varepsilon)\big) \ .$$

But

$$1 = \frac{\mathrm{d}\lambda_k}{\mathrm{d}\varepsilon} = \nabla\lambda_k \cdot \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} = \alpha(\varepsilon)\nabla\lambda_k \cdot \mathbf{t}_k = \alpha(\varepsilon) \ .$$

Thus

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} = \mathbf{t}_k\big(\mathbf{u}(\varepsilon)\big) \quad \text{for } |\varepsilon| < a \ ,$$

and moreover

$$\lambda_k\big(\mathbf{u}(\varepsilon)\big) = \lambda_k(\mathbf{u}_\mathrm{l}) + \varepsilon > \lambda_k(\mathbf{u}_\mathrm{l}) \quad \text{only if } \varepsilon > 0 \ .$$

Having determined the k-integral curve through $\mathbf{u}_\mathrm{l}$, with λ_k varying monotonically along it, we now define the centered k-rarefaction, centered at $(0,0)$, by

$$\begin{cases} \varepsilon = \dfrac{x}{t} - \lambda_k(\mathbf{u}_\mathrm{l}) \ , \\[2ex] \mathbf{u}(x,t) := \mathbf{u}(\varepsilon) \ , \\[2ex] \lambda_k(\mathbf{u}) = \dfrac{x}{t} \ , \end{cases}$$

where $\lambda_k(\mathbf{u}_\mathrm{l})t < x < \big(\lambda_k(\mathbf{u}_\mathrm{l}) + a\big)t$.

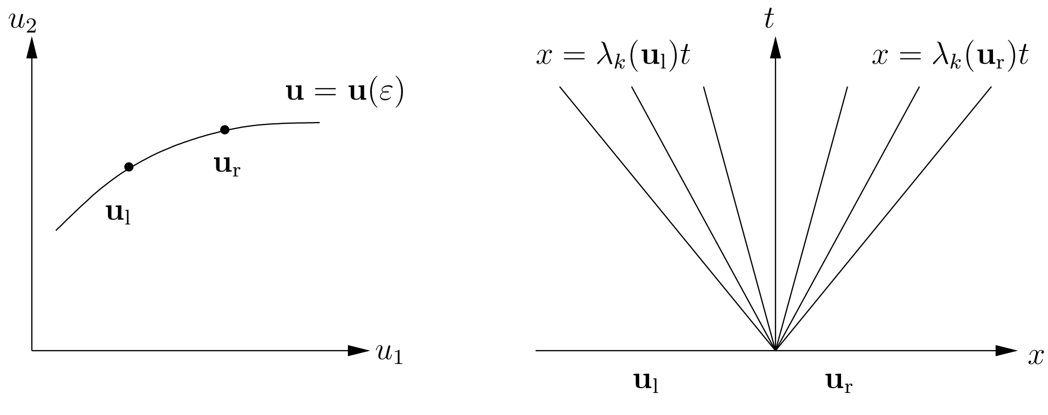


Figure 10.3. *Centered k-rarefaction with* $\mathbf{u}_\mathrm{r}$ *such that* $\lambda_k(\mathbf{u}_\mathrm{l}) < \lambda_k(\mathbf{u}_\mathrm{r}) < \lambda_k(\mathbf{u}_\mathrm{l}) + a$

Taking $\mathbf{u}_\mathrm{r} \in \{\mathbf{u}(\varepsilon) : 0 < \varepsilon < a\}$ and defining the set

$$\mathbb{D} = \{(x,t) \in Q : t > 0,\ \lambda_k(\mathbf{u}_l)t < x < \lambda_k(\mathbf{u}_r)t\} \ ,$$

we obtain that

(*i*) $\mathbf{u} \in C^1(\mathbb{D})$;

(*ii*) $\mathbf{u}$ is a classical solution of (9.1) in $\mathbb{D}$;

(*iii*) all k-Riemann invariants are constant for $\mathbf{u} = \mathbf{u}(x,t)$, with $(x,t) \in \mathbb{D}$,

showing that $\mathbf{u}$ defines indeed a k-rarefaction. We verify (ii): for $(x,t) \in \mathbb{D}$ we compute

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + D\mathbf{f}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} &= -\frac{x}{t^2}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} + \frac{D\mathbf{f}(\mathbf{u})}{t}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} \\ &= \frac{1}{t}\left(-\frac{x}{t}\mathbf{t}_k + D\mathbf{f}(\mathbf{u})\mathbf{t}_k\right) \\ &= \frac{1}{t}\left(-\frac{x}{t} + \lambda_k(\mathbf{u})\right)\mathbf{t}_k = 0 \ .\end{aligned}$$

This completes the proof of the theorem. □

10.3 Shock waves

Having established k-rarefactions in the previous section we now turn to k-shocks.

Definition 10.10. *A k-shock, $1 \leqslant k \leqslant n$, is a pair of states $\{\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}\}$ for which there exists a real number s, called the shock speed, such that*

$$\mathbf{f}(\mathbf{u}_\mathrm{l}) - \mathbf{f}(\mathbf{u}_\mathrm{r}) = s(\mathbf{u}_\mathrm{l} - \mathbf{u}_\mathrm{r}) \tag{10.5}$$

and

$$\lambda_{k-1}(\mathbf{u}_\mathrm{l}) < s < \lambda_k(\mathbf{u}_\mathrm{l}) \ , \tag{10.6a}$$

$$\lambda_k(\mathbf{u}_\mathrm{r}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{r}) \ . \tag{10.6b}$$

Here (10.5) *is the Rankine–Hugoniot shock condition and* (10.6a), (10.6b) *are the Lax entropy conditions. Given any $\mathbf{u}_\mathrm{l} \in N$, we call the set*

$$\{\mathbf{u} \in N : \ \textit{there exists } s \in \mathbb{R} \textit{ such that } \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_\mathrm{l}) = s(\mathbf{u} - \mathbf{u}_\mathrm{l})\}$$

the local Hugoniot locus *for the state $\mathbf{u}_\mathrm{l}$.*

Before investigating the Hugoniot locus, we first recall some notation related to the Taylor expansion in $\mathbb{R}^n$. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be smooth (e.g. $\mathbf{f} \in C^3(\mathbb{R}^n)$). For any pair $\mathbf{u}, \mathbf{u}^0 \in \mathbb{R}^n$ and for $1 \leqslant i \leqslant n$ we have

$$\begin{aligned}f_i(\mathbf{u}) = f_i(\mathbf{u}^0) &+ \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(\mathbf{u}^0)(u_j - u_j^0) \\ &+ \frac{1}{2}\sum_{j,l=1}^{n} \frac{\partial f_i}{\partial u_j \partial u_l}(\mathbf{u}^0)(u_j - u_j^0)(u_l - u_l^0) + \mathcal{O}(||\mathbf{u} - \mathbf{u}^0||^3) \ .\end{aligned} \tag{10.7}$$

Using the symmetric Hessian matrix

$$\big(H(f_i)\big)_{jl} = \frac{\partial^2 f_i}{\partial u_j \partial u_l} \ , \tag{10.8}$$

one writes (10.7) as

$$f_i(\mathbf{u}) = f_i(\mathbf{u}^0) + \nabla f_i(\mathbf{u}^0)\cdot(\mathbf{u}-\mathbf{u}^0) + \frac{1}{2}(\mathbf{u}-\mathbf{u}^0)^T H\big(f_i(\mathbf{u}^0)\big)(\mathbf{u}-\mathbf{u}^0) + \mathcal{O}(||\mathbf{u}-\mathbf{u}^0||^3) \ ,$$

for $1\leqslant i\leqslant n$. Combining all n-components we arrive at the compact notation

$$\mathbf{f}(\mathbf{u}) = \mathbf{f}(\mathbf{u}^0) + A(\mathbf{u}-\mathbf{u}^0) + \frac{1}{2}D^2\mathbf{f}^0(\mathbf{u}-\mathbf{u}^0,\mathbf{u}-\mathbf{u}^0) + \mathcal{O}(||\mathbf{u}-\mathbf{u}^0||^3) \ ,$$

where $A = D\mathbf{f}(\mathbf{u}^0)$, the Jacobian matrix at $\mathbf{u}=\mathbf{u}^0$, and

$$D^2\mathbf{f}^0(\boldsymbol{\xi},\boldsymbol{\eta}) = \Big[\boldsymbol{\xi}^T H\big(f_1(\mathbf{u}^0)\big)\boldsymbol{\eta}, \boldsymbol{\xi}^T H\big(f_2(\mathbf{u}^0)\big)\boldsymbol{\eta}, \cdots, \boldsymbol{\xi}^T H\big(f_n(\mathbf{u}^0)\big)\boldsymbol{\eta}\Big]^T \ .$$

Sometimes we drop the index 0 in this notation.

Next we consider the directional derivative of a vector. Let $\mathbf{e}\in\mathbb{R}^n$. Then we have

$$\frac{\partial f_i}{\partial \mathbf{e}} = \nabla f_i \cdot \mathbf{e} = \big(\mathbf{e}^T\nabla\big) f_i \quad \text{for } i = 1,\cdots,n \ ,$$

or in compact vector notation

$$\frac{\partial \mathbf{f}}{\partial \mathbf{e}} = \big(\mathbf{e}^T\nabla\big)\mathbf{f} \ .$$

Note that in this notation

$$\frac{\partial \mathbf{f}}{\partial \mathbf{f}} = \big(\mathbf{f}^T\nabla\big)\mathbf{f} \neq 1 \ . \qquad (!)$$

We are now in a position to express the condition of genuine nonlinearity (10.4) more directly in terms of the vector function $\mathbf{f}$. We have

Proposition 10.11. *For the k^{th}-characteristic field we have the identity*

$$\frac{\partial \lambda_k}{\partial \mathbf{t}_k} = \nabla\lambda_k \cdot \mathbf{t}_k = \frac{1}{\boldsymbol{\ell}_k^T \mathbf{t}_k} \boldsymbol{\ell}_k^T D^2\mathbf{f}(\mathbf{t}_k,\mathbf{t}_k) \ .$$

Proof. We want to differentiate the equation

$$A\mathbf{t}_k = \lambda_k \mathbf{t}_k \tag{10.9}$$

with respect to $\mathbf{t}_k$. For this we need to evaluate the term $\dfrac{\partial A}{\partial \mathbf{t}_k}\mathbf{t}_k$. This is a vector whose components are given by

$$\begin{aligned}\left(\frac{\partial A}{\partial \mathbf{t}_k}\mathbf{t}_k\right)_i &= \left(\Big(\big(\mathbf{t}_k^T\nabla\big)D\mathbf{f}\Big)\mathbf{t}_k\right)_i = \sum_{m=1}^n \sum_{l=1}^n \left(\Big((t_k)_l \frac{\partial}{\partial u_l}\Big)\frac{\partial f_i}{\partial u_m}\right)(t_k)_m \\ &= \mathbf{t}_k^T H(f_i)\mathbf{t}_k \ .\end{aligned}$$

Thus

$$\frac{\partial A}{\partial \mathbf{t}_k}\mathbf{t}_k = D^2\mathbf{f}(\mathbf{t}_k, \mathbf{t}_k)\ . \tag{10.10}$$

Differentiating (10.9) now gives

$$\frac{\partial A}{\partial \mathbf{t}_k}\mathbf{t}_k + A\frac{\partial \mathbf{t}_k}{\partial \mathbf{t}_k} = \frac{\partial \lambda_k}{\partial \mathbf{t}_k}\mathbf{t}_k + \lambda_k\frac{\partial \mathbf{t}_k}{\partial \mathbf{t}_k}\ .$$

Multiplying on the left by $\boldsymbol{\ell}_k^T$ and using (10.10) gives

$$\boldsymbol{\ell}_k^T D^2\mathbf{f}(\mathbf{t}_k, \mathbf{t}_k) = \frac{\partial \lambda_k}{\partial \mathbf{t}_k}\boldsymbol{\ell}_k^T\mathbf{t}_k\ ,$$

which proves the result. □

Normalization: If the k^{th}-characteristic field is genuinely nonlinear, then $\mathbf{t}_k$ is normalized by (10.4) and, then, $\boldsymbol{\ell}_k$ by $\boldsymbol{\ell}_k^T\mathbf{t}_k = 1$.

The following theorem relates to the existence of a local Hugoniot locus for a given point $\mathbf{u}_\mathrm{l} \in N$.

Theorem 10.12. *Let $\mathbf{u}_\mathrm{l}$ be any point in N. Then there are n smooth one-parameter families of states $\mathbf{u}_k = \mathbf{u}_k(\varepsilon)$, $k = 1, \cdots, n$ and $|\varepsilon| < a_k$, with $\mathbf{u}_k(0) = \mathbf{u}_\mathrm{l}$, all of which satisfy condition* (10.5).

Proof. To prove this result we write

$$\begin{aligned}\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_\mathrm{l}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma}\mathbf{f}\big(\mathbf{u}_\mathrm{l} + \sigma(\mathbf{u} - \mathbf{u}_\mathrm{l})\big)\mathrm{d}\sigma \\ &= \int_0^1 D\mathbf{f}\big(\mathbf{u}_\mathrm{l} + \sigma(\mathbf{u} - \mathbf{u}_\mathrm{l})\big)\mathrm{d}\sigma(\mathbf{u} - \mathbf{u}_\mathrm{l}) \\ &=: \mathcal{G}(\mathbf{u})(\mathbf{u} - \mathbf{u}_\mathrm{l})\ .\end{aligned}$$

The object is to find $\{\mathbf{u}, s\}$ such that (10.5) is satisfied, or equivalently

$$\big[\mathcal{G}(\mathbf{u}) - s\big](\mathbf{u} - \mathbf{u}_\mathrm{l}) = 0\ . \tag{10.11}$$

Since $\lim\limits_{\mathbf{u}\to\mathbf{u}_\mathrm{l}} \mathcal{G}(\mathbf{u}) = D\mathbf{f}(\mathbf{u}_\mathrm{l})$ we have, by continuity, for $\mathbf{u}$ sufficiently close to $\mathbf{u}_\mathrm{l}$, that all eigenvalues of $\mathcal{G}$ are distinct and real:

$$\mu_1(\mathbf{u}) < \mu_2(\mathbf{u}) < \cdots < \mu_n(\mathbf{u})\ .$$

Let $\mathbf{L}_1(\mathbf{u}), \mathbf{L}_2(\mathbf{u}), \cdots, \mathbf{L}_n(\mathbf{u})$ be the corresponding set of left eigenvectors. Now (10.11) has a solution $\{\mathbf{u}, s\}$, with $\mathbf{u} \neq \mathbf{u}_\mathrm{l}$, if and only if $s = \mu_k(\mathbf{u})$ for some k or if and only if $\mathbf{L}_i^T(\mathbf{u})(\mathbf{u} - \mathbf{u}_\mathrm{l}) = 0$, $i = 1, \cdots, n$, $i \neq k$, for some k. The last expression represents $(n-1)$ equations in the n unknowns $\mathbf{u}$. Introducing the $(n-1) \times n$ matrix

$$M(\mathbf{u}) := \begin{bmatrix} \mathbf{L}_1(\mathbf{u}) \\ \vdots \\ \mathbf{L}_k \text{ is missing} \\ \vdots \\ \mathbf{L}_n(\mathbf{u}) \end{bmatrix}\ ,$$

we consider the map $\Phi : N \subseteq \mathbb{R}^n \to \mathbb{R}^{n-1}$ defined by (for some $k \in \{1, \cdots, n\}$)

$$\Phi(\mathbf{u}) = M(\mathbf{u})(\mathbf{u} - \mathbf{u}_l) \ . \tag{10.12}$$

We want to solve $\Phi(\mathbf{u}) = 0$ in some neighbourhood of $\mathbf{u}_l$. Since

$$\begin{aligned}\big(D\Phi(\mathbf{u})\big)_{ij} &= \frac{\partial \Phi_i(\mathbf{u})}{\partial u_j} = \frac{\partial}{\partial u_j}\left\{\sum_{p=1}^{n} M_{ip}(\mathbf{u})(u_p - u_{lp})\right\} \\ &= \sum_{p=1}^{n} \frac{\partial M_{ip}(\mathbf{u})}{\partial u_j}(u_p - u_{lp}) + M_{ij}(\mathbf{u}) \ ,\end{aligned}$$

we see that

$$D\Phi(\mathbf{u}_l) = M(\mathbf{u}_l) \ ,$$

which has rank $(n-1)$. Hence there must exist an $(n-1) \times (n-1)$ nonsingular minor and the implicit function theorem then tells us that (10.12) defines a curve $\mathbf{u}_k = \mathbf{u}_k(\varepsilon)$ as stated in the theorem. This proof leads to n curves $k = 1, \cdots, n$ through the point $\mathbf{u}_l$. The curves are all distinct near $\mathbf{u}_l$ since

$$\mathbf{u} - \mathbf{u}_l \perp \operatorname{span}\{\mathbf{L}_1(\mathbf{u}), \cdots, \mathbf{L}_k \text{ missing }, \cdots, \mathbf{L}_n(\mathbf{u})\}$$

for each k. □

As to be expected we have

Proposition 10.13. $\dot{\mathbf{u}}_k(0) = c_k \mathbf{t}_k(\mathbf{u}_l)$ *for some* $c_k \neq 0$.

Proof. Dividing the expression

$$\mathbf{f}\big(\mathbf{u}_k(\varepsilon)\big) - \mathbf{f}(\mathbf{u}_l) = \mu_k\big(\mathbf{u}_k(\varepsilon)\big)\big(\mathbf{u}_k(\varepsilon) - \mathbf{u}_l\big)$$

by ε and passing to the limit for $\varepsilon \to 0$, we find

$$D\mathbf{f}(\mathbf{u}_l)\dot{\mathbf{u}}_k(0) = \lambda_k(\mathbf{u}_l)\dot{\mathbf{u}}_k(0) \ .$$

□

The next result is concerned with the parametrization of the Hugoniot locus.

Proposition 10.14. *Along the k-shock (i.e. along the curve* $\mathbf{u} = \mathbf{u}_k$*), if the* k^{th}*-characteristic field is genuinely nonlinear, we can choose a parametrization so that* $\dot{\mathbf{u}}_k(0) = \mathbf{t}_k(0)$ *and* $\ddot{\mathbf{u}}_k(0) = \dot{\mathbf{t}}_k(0)$*. Moreover* $s(0) = \lambda_k(\mathbf{u}_l)$ *and* $\dot{s}(0) = \frac{1}{2}$*. Here* $\mathbf{t}_k(\varepsilon) := \mathbf{t}_k\big(\mathbf{u}_k(\varepsilon)\big)$ *and* $s(\varepsilon) := s\big(\mathbf{u}_k(\varepsilon)\big)$*.*

Proof. Given any parametrization (as a result of Theorem 10.12), we redefine ε with $\varepsilon := c_k\varepsilon$, yielding

$$\dot{\mathbf{u}}_k(0) = \mathbf{t}_k(0) \ . \tag{10.13}$$

Writing also $\lambda_k(\varepsilon) := \lambda_k\big(\mathbf{u}_k(\varepsilon)\big)$ we obtain from (10.4) and (10.13)

$$\dot{\lambda}_k(0) = \nabla\lambda_k(\mathbf{u}_l) \cdot \dot{\mathbf{u}}_k(0) = \nabla\lambda_k(\mathbf{u}_l) \cdot \mathbf{t}_k(\mathbf{u}_l) = 1 \ .$$

Next consider the expression

$$D\mathbf{f}\big(\mathbf{u}_k(\varepsilon)\big)\mathbf{t}_k(\varepsilon) = \lambda_k(\varepsilon)\mathbf{t}_k(\varepsilon) \ .$$

Differentiating with respect to ε and setting $\varepsilon = 0$ gives

$$D^2\mathbf{f}(\mathbf{t}_k, \mathbf{t}_k) + D\mathbf{f}\dot{\mathbf{u}}_k = \mathbf{t}_k + \lambda_k\dot{\mathbf{t}}_k \ . \tag{10.14}$$

Differentiating the shock condition (10.5) with respect to ε yields

$$D\mathbf{f}\dot{\mathbf{u}}_k = \dot{s}(\mathbf{u}_k - \mathbf{u}_\mathrm{l}) + s\dot{\mathbf{u}}_k \quad (\Rightarrow s(0) = \lambda_k(\mathbf{u}_\mathrm{l}))$$

and again

$$D^2\mathbf{f}(\dot{\mathbf{u}}_k, \dot{\mathbf{u}}_k) + D\mathbf{f}\ddot{\mathbf{u}}_k = \ddot{s}(\mathbf{u}_k - \mathbf{u}_\mathrm{l}) + 2\dot{s}\dot{\mathbf{u}}_k + s\ddot{\mathbf{u}}_k \ .$$

At $\varepsilon = 0$ this becomes

$$D^2\mathbf{f}(\mathbf{t}_k, \mathbf{t}_k) + D\mathbf{f}\ddot{\mathbf{u}}_k = 2\dot{s}\mathbf{t}_k + \lambda_k\ddot{\mathbf{u}}_k \ . \tag{10.15}$$

Substracting (10.14) from (10.15) results in

$$D\mathbf{f}(\ddot{\mathbf{u}}_k - \dot{\mathbf{t}}_k) = (1 - 2\dot{s})\mathbf{t}_k + \lambda_k(\ddot{\mathbf{u}}_k - \dot{\mathbf{t}}_k) \ ,$$

or

$$(D\mathbf{f} - \lambda_k)(\ddot{\mathbf{u}}_k - \dot{\mathbf{t}}_k) = (1 - 2\dot{s})\mathbf{t}_k \ .$$

Multiplying on the left by $\boldsymbol{\ell}_k^T$ gives

$$1 - 2\dot{s} = 1 \quad \Rightarrow \quad \dot{s}(0) = \frac{1}{2}$$

and thus

$$\ddot{\mathbf{u}}_k(0) - \dot{\mathbf{t}}_k(0) = c\,\mathbf{t}_k(0) \quad \text{for some } c \in \mathbb{R} \ . \tag{10.16}$$

Again we redefine the parametrization

$$\varepsilon = \delta - \frac{1}{2}c\delta^2 \ .$$

This yields

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\delta} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\delta} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon}(1 - c\delta) \quad \Rightarrow \quad \left.\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\delta}\right|_{\delta=0} = \left.\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}$$

and

$$\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}\delta^2} = \frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}\varepsilon^2}(1 - c\delta)^2 - c\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} \quad \Rightarrow \quad \left.\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}\delta^2}\right|_{\delta=0} = \left.\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=0} - c\left.\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} .$$

Hence in terms of δ we obtain from (10.16)

$$\ddot{\mathbf{u}}_k(0) - \dot{\mathbf{t}}_k(0) = 0 \ .$$

$\square$

We can now conclude

Theorem 10.15. *The Lax entropy inequalities* (10.6a), (10.6b) *hold along the k-shock* $\mathbf{u}_k(\varepsilon)$*, if and only if* $\varepsilon < 0$*.*

Proof. In the notation of Proposition 10.14 the entropy conditions are

$$\begin{aligned} &(i) \quad \lambda_{k-1}(0) < s(\varepsilon) < \lambda_k(0) \ , \\ &(ii) \quad \lambda_k(\varepsilon) < s(\varepsilon) < \lambda_{k+1}(\varepsilon) \ . \end{aligned}$$

Let $w(\varepsilon) := \lambda_k(\varepsilon) - s(\varepsilon)$. Then $w(0) = 0$ and $w'(0) = 1 - \frac{1}{2} > 0$. Thus if (ii) holds then $\varepsilon < 0$. Conversely if $\varepsilon < 0$, then $w(\varepsilon) < 0$ implying $\lambda_k(\varepsilon) < s(\varepsilon)$. Moreover $\dot{s}(0) = \frac{1}{2}$ and $\lambda_k(0) = s(0)$ imply $s(\varepsilon) < \lambda_k(0)$. Since $s(\varepsilon) \to \lambda_k(0) > \lambda_{k-1}(0)$ as $\varepsilon \to 0$ we have $s(\varepsilon) > \lambda_{k-1}(0)$ for ε small. Finally, since $\lambda_{k+1}(0) > \lambda_k(0) = s(0)$ we also have $\lambda_{k+1}(\varepsilon) > s(\varepsilon)$ for ε small. □

Combining Theorem 10.9 and the results of this section we obtain for any $\mathbf{u}_\mathrm{l} \in N$ the existence of n composite curves $(1 \leqslant k \leqslant n)$,

$$\mathbf{U}_k = \begin{cases} \mathbf{u}_{k,\mathrm{R}}(\varepsilon) & \text{for } \varepsilon \geqslant 0 \ , \\ \mathbf{u}_{k,\mathrm{S}}(\varepsilon) & \text{for } \varepsilon \leqslant 0 \ , \end{cases}$$

where $\mathbf{u}_{k,\mathrm{R}}$ is the k-rarefaction curve and $\mathbf{u}_{k,\mathrm{S}}$ the k-shock curve. These curves are twice continuously differentiable in ε.

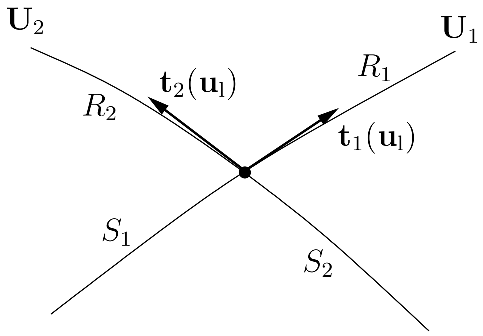


Figure 10.4. *Example of composite curves* ($n = 2$)

In some physical systems, for instance the Euler equations, it happens that one of the characteristic fields satisfies

$$\nabla\lambda_k(\mathbf{u}) \cdot \mathbf{t}_k(\mathbf{u}) = 0 \quad \text{for all } \mathbf{u} \in N \ .$$

Then the k^{th} characteristic field is called *linearly degenerate*. By definition, λ_k is now a Riemann-invariant. It has the following consequence. Consider the initial value problem

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\varepsilon} = \mathbf{t}_k\big(\mathbf{u}(\varepsilon)\big) \ , \quad \mathbf{u}(0) = \mathbf{u}_\mathrm{l} \ .$$

This problem has a unique solution $\mathbf{u}(\varepsilon)\in N$ for $|\varepsilon|<a$. Moreover, $\lambda_k\big(\mathbf{u}(\varepsilon)\big)=\text{constant}=\lambda_k(\mathbf{u}_\mathrm{l})$ for all $|\varepsilon|<a$. Now, for $|\varepsilon|<a$, consider the Riemann problem

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}+\dfrac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=0 & \text{in } Q\ , \\ \\ \mathbf{u}(x,0)=\begin{cases}\mathbf{u}_\mathrm{l} & x<0\ ,\\ \mathbf{u}(\varepsilon) & x>0\ .\end{cases} \end{cases}$$

It's solution is given by

$$\mathbf{u}(x,t)=\begin{cases}\mathbf{u}_\mathrm{l} & \text{for } x<t\lambda_k(\mathbf{u}_\mathrm{l})\ ,\\ \mathbf{u}(\varepsilon) & \text{for } x>t\lambda_k(\mathbf{u}_\mathrm{l})\ .\end{cases}$$

Indeed, checking the Rankine–Hugoniot conditions gives (with $s=\lambda_k(\mathbf{u}_\mathrm{l})$)

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big\{\mathbf{f}\big(\mathbf{u}(\varepsilon)\big)-s\mathbf{u}(\varepsilon)\big\}=D\mathbf{f}\dot{\mathbf{u}}-s\dot{\mathbf{u}}=(D\mathbf{f}-\lambda_k I)\mathbf{t}_k=(\lambda_k\big(\mathbf{u}(\varepsilon)\big)-\lambda_k(\mathbf{u}_\mathrm{l}))\mathbf{t}_k=0\ .$$

Thus

$$\mathbf{f}\big(\mathbf{u}(\varepsilon)\big)-s\mathbf{u}(\varepsilon)=\mathbf{f}(\mathbf{u}_\mathrm{l})-s\mathbf{u}_\mathrm{l}$$

and consequently

$$\mathbf{f}\big(\mathbf{u}(\varepsilon)\big)-\mathbf{f}(\mathbf{u}_\mathrm{l})=s\big(\mathbf{u}(\varepsilon)-\mathbf{u}_\mathrm{l}\big)\qquad \text{for all } |\varepsilon|<a\ , \tag{10.17}$$

showing that all points $\mathbf{u}(\varepsilon)$ can be connected to $\mathbf{u}_\mathrm{l}$ by a shock of the same speed $s=\lambda_k(\mathbf{u}_\mathrm{l})$. Such a solution (or shock) is called a *contact-discontinuity*.

We have the following result

Theorem 10.16. *If two nearby states $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ have the same k-Riemann invariants with respect to a linearly degenerate field, then they are connected to each other by a contact discontinuity of speed $s=\lambda_k(\mathbf{u}_\mathrm{l})=\lambda_k(\mathbf{u}_\mathrm{r})$.*

Proof. Because $w_i(\mathbf{u}_\mathrm{l})=w_i(\mathbf{u}_\mathrm{r})$ for $i=1,\cdots,n-1$, it follows that $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ belong to the same k-integral curve, along which λ_k is constant (and therefore the curve is also a k^{th}-characteristic curve), and along which (10.17) holds. Thus $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ can be connected by a contact-discontinuity. □

10.4 Solvability

We are now in a position to give the main result of this section. Consider the Riemann problem

$$(R)\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}+\dfrac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=0 & \text{in } Q\ , \\ \\ \mathbf{u}(x,0)=\begin{cases}\mathbf{u}_\mathrm{l} & x<0\ ,\\ \mathbf{u}_\mathrm{r} & x>0\ .\end{cases} \end{cases}$$

where $\mathbf{f}: N \subset \mathbb{R}^n \to \mathbb{R}^n$ is smooth and strictly hyperbolic, such that either $\nabla\lambda_k \cdot \mathbf{t}_k = 1$ or $\nabla\lambda_k \cdot \mathbf{t}_k = 0$ in N for $1 \leqslant k \leqslant n$. Then combining the composite curves $\mathbf{U}_k(\varepsilon)$, provided the k^{th}-characteristic fields are genuinely nonlinear, and k-integral curves, for which the corresponding characteristic fields are linearly degenerate, we obtain a unique solution of Problem R provided $||\mathbf{u}_\mathrm{l} - \mathbf{u}_\mathrm{r}||$ is small. This solution consists of at most $(n+1)$ constant states, separated by shocks, centered rarefactions or contact-discontinuities. The proof of this fundamental existence and uniqueness result combines the theory developed in this chapter. Below we give the main idea.

For each $k = 1, 2, \cdots, n$ consider the family of transformations

$$\mathbf{T}^k_{\varepsilon_k} : N \to \mathbb{R}^n \ , \quad |\varepsilon_k| < a \ ,$$

where $\mathbf{T}^k_{\varepsilon_k}(\mathbf{u}_\mathrm{l})$, with $\mathbf{u}_\mathrm{l} \in N$, is the admissible right state for $\mathbf{u}_\mathrm{l}$ (admissible shock, rarefaction or contact-discontinuity). Thus $\mathbf{T}^k_{\varepsilon_k}(\mathbf{u}_\mathrm{l})$ is the composite curve $\mathbf{U}_k(\varepsilon_k)$ if the k^{th}-field is genuinely nonlinear, or the k^{th}-characteristic curve if the k^{th}-field is linearly degenerate. Let C denote the hypercube

$$C = \{(\varepsilon_1, \cdots, \varepsilon_n) \in \mathbb{R}^n : |\varepsilon_i| < a \text{ for } i = 1, 2, \cdots, n\} \ .$$

Next consider the composite transformation $\mathbf{T}: C \to \mathbb{R}^n$ given by

$$\mathbf{T}(\boldsymbol{\varepsilon}) = \mathbf{T}^n_{\varepsilon_n} \mathbf{T}^{n-1}_{\varepsilon_{n-1}} \cdots \mathbf{T}^1_{\varepsilon_1}(\mathbf{u}_\mathrm{l}) \qquad (\mathbf{u}_\mathrm{l} \in N \text{ fixed}) \ ,$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$. We want to show that there exists a unique $\boldsymbol{\varepsilon}^* \in C$ such that

$$\mathbf{T}(\boldsymbol{\varepsilon}^*) = \mathbf{u}_\mathrm{r} \in N \ ,$$

when $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ are sufficiently close. To achieve this we introduce $\mathbf{F}: C \to \mathbb{R}^n$ given by

$$\mathbf{F}(\boldsymbol{\varepsilon}) = \mathbf{T}(\boldsymbol{\varepsilon}) - \mathbf{u}_\mathrm{l} \ .$$

Then $\mathbf{F}(0) = 0$, and since

$$\mathbf{T}^k_{\varepsilon_k}(\mathbf{u}) = \mathbf{u} + \varepsilon_k \mathbf{t}_k(\mathbf{u}) + \mathcal{O}(\varepsilon_k^2) \ ,$$

we have

$$\mathbf{F}(\boldsymbol{\varepsilon}) = \sum_{k=1}^{n} \varepsilon_k \mathbf{t}_k + \mathcal{O}(||\boldsymbol{\varepsilon}||^2) \ .$$

Hence $\mathbf{F}$ maps a neighbourhood of $\varepsilon = 0$ one-to-one onto a neighbourhood of $\mathbf{u} = 0$. Thus if $||\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l}||$ is sufficiently small, there exists a unique $\boldsymbol{\varepsilon}^* \in C$ such that

$$\mathbf{F}(\boldsymbol{\varepsilon}^*) = \mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l} \ .$$

In other words,

$$\mathbf{T}(\boldsymbol{\varepsilon}^*) = \mathbf{u}_\mathrm{r} \ ,$$

which proves the assertion.

Remark 10.17. *The results obtained in this chapter are* local *results, because we used (in essence) only smoothness of the flux* $\mathbf{f}$ *and no structural properties. In many applications, however, global solutions exist (see for instance example 9.1) without the smallness restriction on* $||\mathbf{u}_\mathrm{r} - \mathbf{u}_\mathrm{l}||$.

11 Entropy and viscous profiles

In this chapter we introduce an entropy for the hyperbolic system as an alternative concept to provide uniqueness. In particular we demonstrate equivalence between the Lax entropy (or shock) conditions (10.6) and an entropy inequality across the shock. Further we consider travelling waves of the viscous perturbation (i.e. viscous profiles) and discuss their existence in terms of the Lax inequalities.

11.1 Entropy inequality

In Chapter 7 we presented the equations of gas dynamics in terms of density, velocity and entropy. Following SMOLLER [67], we now take internal energy as one of the dependent variables and we use Lagrangian coordinates. Then there results

$$
\begin{aligned}
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} &= 0 \quad \text{(conservation of mass)} \; ; \\
\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} &= 0 \quad \text{(conservation of momentum)} \; ; \\
\frac{\partial}{\partial t}(e + \frac{1}{2}u^2) + \frac{\partial}{\partial x}(pu) &= 0 \quad \text{(conservation of energy)} \; ,
\end{aligned}
$$

where $v = \rho^{-1}$ is the specific volume, $\rho =$density, $u =$velocity, $p =$pressure and $e =$internal energy. Using these equations and the second law of thermodynamics, one recovers for the entropy S the additional conservation equation

$$
\frac{\partial S}{\partial t} = 0 \; . \tag{11.1}
$$

Note that the derivation of (11.1) is purely formal: it applies only when considering smooth solutions in the absence of shocks. When shocks are present, we need to interpret (11.1) in a weak sence and find

$$
\frac{\partial S}{\partial t} \geqslant 0 \quad (\text{or } \int_Q S\frac{\partial \varphi}{\partial t} \leqslant 0 \text{ for all } \varphi \in C_0^\infty(Q) \; , \;\; \varphi \geqslant 0) \; , \tag{11.2}
$$

implying that the entropy of a fluid particle increases when going through a shock. This entropy inequality plays a crucial role when selecting the physically correct shock solution.

This concept has been generalized in the mathematics literature. Mathematicians, however, like to see quantities stabilize as time increases. Therefore, considering $-S$, we ask ourselves the question: when does a system of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \quad \text{in } Q \tag{11.3}$$

imply the existence of an additional conservation law

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad (\text{or } \leqslant \text{ weakly}) \text{ in } Q\ , \tag{11.4}$$

where $U = U(u_1, \cdots, u_n)$ and $F = F(u_1, \cdots, u_n)$? If it does then we call U an *entropy* and F the corresponding entropy flux for (11.3), see also LAX [45] or SMOLLER [67]. For $n = 1$ (scalar case) this is precisely the Kruzkov formulation given in Section 6.3. Then there are infinitely many entropy functions, one for each $k \in \mathbb{R}$. To answer this question for $n \geqslant 2$ we carry out the differentiation in (11.4)

$$(\nabla_u U)^T \frac{\partial \mathbf{u}}{\partial t} + (\nabla_u F)^T \frac{\partial \mathbf{u}}{\partial x} = 0$$

and we multiply (11.3) on the left by $\nabla_u U$ giving

$$(\nabla_u U)^T \frac{\partial \mathbf{u}}{\partial t} + (\nabla_u U)^T D\mathbf{f} \frac{\partial \mathbf{u}}{\partial x} = 0\ .$$

Thus (11.3) has an entropy U, if and only if we can find a pair (U, F) satisfying

$$(\nabla_u U)^T D\mathbf{f} = (\nabla_u F)^T\ . \tag{11.5}$$

This is a system of n partial differential equations for the two unknowns U and F. Thus if $n > 2$, this system is overdetermined and usually has no solutions. However, there are some important cases for which a nontrivial solution exists.

Example 11.1. *Suppose* $\mathbf{f}$ *is a gradient, that is there exists a function* $\phi(\mathbf{u})$ *such that* $\mathbf{f}(\mathbf{u}) = \nabla_u \phi(\mathbf{u})$. *We set*

$$U(\mathbf{u}) = \frac{1}{2}||\mathbf{u}||^2 \quad \textit{and} \quad F(\mathbf{u}) = \mathbf{u} \cdot \mathbf{f}(\mathbf{u}) - \phi(\mathbf{u})\ .$$

Then

$$\begin{aligned} \frac{\partial F}{\partial x} &= \left(\frac{\partial \mathbf{u}}{\partial x} \cdot \mathbf{f}(\mathbf{u})\right) + \left(\mathbf{u} \cdot \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}\right) - \left(\nabla_u \phi \cdot \frac{\partial \mathbf{u}}{\partial x}\right) \\ &= \left(\mathbf{u} \cdot \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}\right) = -\left(\mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t}\right) = -\frac{\partial U}{\partial t}\ . \end{aligned}$$

Example 11.2. *Let* $n = 2$ *and consider the antigradient system*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\phi_v(u, v) = 0\ , \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}\phi_u(u, v) = 0\ . \end{cases}$$

Then $U=\phi$ and $F=\phi_u\phi_v$. Indeed,

$$\frac{\partial U}{\partial t}=\phi_u\frac{\partial u}{\partial t}+\phi_v\frac{\partial v}{\partial t}=-\phi_u\frac{\partial \phi_v}{\partial x}-\phi_v\frac{\partial \phi_u}{\partial x}=-\frac{\partial}{\partial x}\left(\phi_u\phi_v\right) \ .$$

Note that systems of the form (compare the p-system)

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial f(v)}{\partial x}=0\\[2mm] \dfrac{\partial v}{\partial t}+\dfrac{\partial g(u)}{\partial x}=0\end{cases}$$

are special cases of antigradient systems with

$$\phi(u,v)=\int^v f(s)ds+\int^u g(s)ds \ .$$

The role of the entropy conditions (10.6) is to obtain the physically relevant solution from all the others. This can also be done by considering the method of vanishing viscosity as in Chapter 6. In this method we perturb equations (11.3) and study

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=\varepsilon\frac{\partial^2\mathbf{u}}{\partial x^2} \quad \text{in } Q \ , \tag{11.6}$$

where $\varepsilon>0$.
Now suppose (11.3) has an entropy U and entropy flux F satisfying (11.5). Further suppose that U is convex: i.e. the Hessian $H(U)$ is positive definite. Then multiplying (11.6) on the left by $\nabla_u U$ gives

$$\begin{aligned}\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}&=\varepsilon(\nabla_u U)^T\frac{\partial^2\mathbf{u}}{\partial x^2}\\ &=\varepsilon\bigg((\nabla_u U)^T\frac{\partial \mathbf{u}}{\partial x}\bigg)_x-\varepsilon\bigg(\frac{\partial \mathbf{u}}{\partial x}\bigg)^T H(U)\frac{\partial \mathbf{u}}{\partial x}\\ &\leqslant \varepsilon U_{xx} \ .\end{aligned}$$

We multiply this inequality by $\varphi\in C_0^\infty(Q)$, $\varphi\geqslant 0$ and integrate the result over Q to obtain

$$\int_Q\{U\varphi_t+F\varphi_x\}\geqslant -\varepsilon\int_Q U\varphi_{xx} \ .$$

Next we assume that $||\mathbf{u}||$, and thus $U(\mathbf{u})$, is uniformly bounded with respect to ε. Then, as $\varepsilon\to 0$,

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}\leqslant 0 \tag{11.7}$$

in sense of distributions. Thus we have shown

Theorem 11.3. *Suppose there exist an entropy pair (U,F) related to the system* (11.3). *Let* $\mathbf{u}$ *be a weak solution of* (11.3) *which is the weak limit of uniformly bounded solutions of the viscosity equation* (11.6). *If U is convex, then* $\mathbf{u}$ *satisfies* (11.7) *in sense of distributions.*

Note that the inequality in (11.7) appears as a result of the regularization, as it did in the scalar case in Section 6. Similarly we have

Corollary 11.4. *Suppose Theorem 11.3 holds. If* $\mathbf{u}$ *is a piecewise smooth solution, then accross each discontinuity* $\mathbf{u}$ *satisfies*

$$s\big[U(\mathbf{u}_\mathrm{r}) - U(\mathbf{u}_\mathrm{l})\big] \geqslant F(\mathbf{u}_\mathrm{r}) - F(\mathbf{u}_\mathrm{l}) \;, \tag{11.8}$$

where s *is the speed of the discontinuity, and* $\mathbf{u}_\mathrm{r}$ *and* $\mathbf{u}_\mathrm{l}$ *are, respectively, the right and left state at the discontinuity.*

The following theorem relates the Lax entropy inequalities (10.6) and inequality (11.8).

Theorem 11.5. *Suppose system* (11.3) *is hyperbolic and genuinely nonlinear in some neighbourhood* $N \subset \mathbb{R}^n$. *Further, suppose that* (11.3) *admits an entropy pair* (U, F) *where* U *is strictly convex. Finally, let* $\mathbf{u}$ *be a piecewise smooth solution of* (11.3) *which is the weak limit of uniformly bounded solutions of* (11.6). *Accross any k-shock in* N *we have: the Lax entropy inequalities* (10.6) *hold if and only if* (11.8) *holds.*

Proof. We use Theorem 10.15 and set $\mathbf{u} = \mathbf{u}(\varepsilon) = \mathbf{u}_k(\varepsilon)$. Let $\mathbf{u}_\mathrm{l} \in N$. The Lax entropy inequalities hold, with $\mathbf{u}(\varepsilon)$ as right state, if and only if $\varepsilon < 0$. For $\overline{\varepsilon} < \varepsilon < 0$ ($|\overline{\varepsilon}|$ sufficiently small), set

$$I(\varepsilon) = s(\varepsilon)\big[U(\varepsilon) - U_\mathrm{l}\big] - \big[F(\varepsilon) - F_\mathrm{l}\big] \;, \tag{11.9}$$

where $U(\varepsilon),\ F(\varepsilon) = U(\mathbf{u}(\varepsilon)),\ F(\mathbf{u}(\varepsilon))$ and $U_\mathrm{l},\ F_\mathrm{l} = U(\mathbf{u}_\mathrm{l}),\ F(\mathbf{u}_\mathrm{l})$. Further, $s(\varepsilon)$ denotes the shock speed of the discontinuity $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r} = \mathbf{u}(\varepsilon)$. To prove the theorem we need to show $I(\varepsilon) > 0$ if and only if $\varepsilon < 0$. Differentiating (11.9) gives

$$\dot{I} = \dot{s}\big[U - U_\mathrm{l}\big] - s\big(\nabla_u U\big)^T \dot{\mathbf{u}} - \big(\nabla_u F\big)^T \dot{\mathbf{u}} \;,$$

and using (11.5)

$$\dot{I} = \dot{s}\big[U - U_\mathrm{l}\big] + s\big(\nabla_u U\big)^T \dot{\mathbf{u}} - \big(\nabla_u U\big)^T D\mathbf{f}\,\dot{\mathbf{u}} \;.$$

Since $s(\mathbf{u} - \mathbf{u}_\mathrm{l}) = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_\mathrm{l})$, we have

$$D\mathbf{f}\,\dot{\mathbf{u}} = \dot{s}(\mathbf{u} - \mathbf{u}_\mathrm{l}) + s\dot{\mathbf{u}} \;.$$

Consequently,

$$\dot{I} = \dot{s}\big[U - U_\mathrm{l}\big] - \dot{s}\big(\nabla_u U\big)^T (\mathbf{u} - \mathbf{u}_\mathrm{l}) \;,$$

implying $\dot{I}(0) = 0$. Hence we need to determine the second derivative at $\varepsilon = 0$. Following SMOLLER [67] and using Proposition 10.14 we find $\ddot{I}(0) = 0$ as well. Differentiating again gives

$$\dddot{I}(0) = -\dot{s}(\dot{\mathbf{u}})^T H(U)\dot{\mathbf{u}}\Big|_{\varepsilon=0} = -\frac{1}{2}(\mathbf{t}_k)^T H(U)\mathbf{t}_k\Big|_{\mathbf{u}_\mathrm{l}} < 0 \tag{11.10}$$

by the strict convexity of U. Hence $I(\varepsilon) > 0$ if and only if $\varepsilon < 0$, for $|\varepsilon|$ sufficiently small. □

So for weak shocks (i.e. $||\mathbf{u}_\mathrm{l} - \mathbf{u}_\mathrm{r}||$ sufficiently small) we have established equivlence between the Lax conditions (10.6) and the entropy formulation (11.7), (11.8), provided to hyperbolic system is genuinely nonlinear.

Remark 11.6. *If we replace the viscosity equation* (11.6) *by*

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = \varepsilon A \frac{\partial^2 \mathbf{u}}{\partial x^2} , \tag{11.11}$$

where A is an $n \times n$ positive semidefinite matrix with constant coefficients, and if an entropy pair (U, F) exists (i.e. $\nabla_u U D\mathbf{f} = \nabla_u F$), then the conclusion of Theorem 11.5 remains valid, provided

$$H(U)A \geqslant cI , \tag{11.12}$$

where c is a positive constant.

11.2 Viscous profiles

As in the scalar case, travelling waves can be used to distinguish the physically correct shock from all the others. When they exist, we say that the shock has a viscous profile or structure.

Again we consider system (11.3) for which we assume

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_n(\mathbf{u}) .$$

Let $(\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}, s)$ denote a k-shock satisfying the Lax inequalities. Then

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_\mathrm{l} & \text{as } x < st , \\ \mathbf{u}_\mathrm{r} & \text{as } x > st , \end{cases} \tag{11.13}$$

where

$$\mathbf{f}(\mathbf{u}_\mathrm{l}) - \mathbf{f}(\mathbf{u}_\mathrm{r}) = s(\mathbf{u}_\mathrm{l} - \mathbf{u}_\mathrm{r}) \tag{11.14}$$

and

$$\lambda_1(\mathbf{u}_\mathrm{l}) < \cdots < \lambda_{k-1}(\mathbf{u}_\mathrm{l}) < s < \lambda_k(\mathbf{u}_\mathrm{l}) < \cdots < \lambda_n(\mathbf{u}_\mathrm{l}) , \tag{11.15a}$$
$$\lambda_1(\mathbf{u}_\mathrm{r}) < \cdots < \lambda_k(\mathbf{u}_\mathrm{r}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{r}) < \cdots < \lambda_n(\mathbf{u}_\mathrm{r}) . \tag{11.15b}$$

Has this k-shock a viscous structure? That is, given a certain parabolic regularization of (11.3), is there a travelling wave that collapses onto the shock as the small parameter vanishes? Let us consider the simplest regularization possible. Adding a linear viscosity term to (11.3) gives

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = \varepsilon \frac{\partial^2 \mathbf{u}}{\partial x^2} \quad \text{in } Q . \tag{11.16}$$

Considering the travelling wave

$$\mathbf{u}(x,t) = \mathbf{u}(\eta) \quad \text{with } \eta = \frac{x - st}{\varepsilon} ,$$

we find that $\mathbf{u}$ should satisfy the system of ordinary differential equations

$$-s\mathbf{u}' + \big(\mathbf{f}(\mathbf{u})\big)' = \mathbf{u}'' . \tag{11.17}$$

Here the primes denote differentiation with respect to η. This equation can be integrated to give

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}) - s\mathbf{u} + C\ , \tag{11.18}$$

where C is the constant of integration. If $\mathbf{u}(\eta)$ is to converge towards the k-shock (11.13) as $\varepsilon \downarrow 0$, we need to consider (11.18) for all $\eta \in \mathbb{R}$, subject to the boundary condition

$$\mathbf{u}(-\infty) = \mathbf{u}_\mathrm{l} \quad \text{and} \quad \mathbf{u}(+\infty) = \mathbf{u}_\mathrm{r}\ .$$

Hence $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ must be rest points for (11.18). This gives

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_\mathrm{l}) - s(\mathbf{u} - \mathbf{u}_\mathrm{l}) \quad \text{in } \mathbb{R}\ , \tag{11.19}$$

with s satisfying the Rankine-Hugoniot conditions (11.15). Thus the speed of the travelling wave (if it exists) and the shock speed coincide.

Following COURANT & FRIEDRICHS [16], HOPF [37] and GELFAND [26], we call the shock (11.13) admissible if it has a viscous structure or profile: i.e. if (11.19) has a solution in $\mathbb{R}^n$ connecting the rest points $\mathbf{u}_\mathrm{l}$ (as $\eta \to -\infty$) and $\mathbf{u}_\mathrm{r}$ (as $\eta \to +\infty$). We discuss below that if the Lax conditions (11.15) are satisfied, the k-shock is admissible. For this purpose we investigate the nature of the dynamical system (11.19) at $\mathbf{u} = \mathbf{u}_\mathrm{l}$ and $\mathbf{u} = \mathbf{u}_\mathrm{r}$. Linearization gives the eigenvalues

$$\begin{aligned} e_k(\mathbf{u}_\mathrm{l}) &= \lambda_k(\mathbf{u}_\mathrm{l}) - s\ ,\\ e_k(\mathbf{u}_\mathrm{r}) &= \lambda_k(\mathbf{u}_\mathrm{r}) - s\ . \end{aligned}$$

Thus if (11.15) is satisfied, then

$$e_1(\mathbf{u}_\mathrm{l}) < \cdots < e_{k-1}(\mathbf{u}_\mathrm{l}) < 0 < e_k(\mathbf{u}_\mathrm{l}) < \cdots < e_k(\mathbf{u}_\mathrm{l})\ , \tag{11.20a}$$

$$e_1(\mathbf{u}_\mathrm{r}) < \cdots < e_k(\mathbf{u}_\mathrm{r}) < 0 < e_{k+1}(\mathbf{u}_\mathrm{r}) < \cdots < e_k(\mathbf{u}_\mathrm{r})\ . \tag{11.20b}$$

Since the number of positive (and different) eigenvalues at a rest point gives the dimension of the unstable manifold, and the number of negative eigenvalues the dimension of the stable manifold we have:

$$\dim(\text{ unstable manifold at } \mathbf{u}_\mathrm{l}\ ) + \dim(\text{ stable manifold at } \mathbf{u}_\mathrm{r}\ ) = n - k + 1 + k = n + 1 > n$$

i.e. the sum exceeds the dimension of the space in which (11.19) is solved. Intuitively one expects that this guarantees the existence of a connecting orbit flowing from $\mathbf{u}_\mathrm{l}$ to $\mathbf{u}_\mathrm{r}$, and that this connection is stable under small perturbations of $\mathbf{u}_\mathrm{l}$, $\mathbf{u}_\mathrm{r}$ and of the viscosity matrix (chosen here as the identity). Indeed, using geometrical arguments and the Conley index, SMOLLER [67] proved this rigorously for $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$ sufficiently close.

We now consider the system

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(D(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}\right) \quad \text{in } Q\ , \tag{11.21}$$

where $D(\mathbf{u})$ is the viscosity matrix modelling those physical effects that are disregarded in the conservation laws. Putting again $\mathbf{u} = \mathbf{u}(\eta)$, we obtain as in (11.19)

$$\begin{cases} D(\mathbf{u})\mathbf{u}' = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_\mathrm{l}) - s(\mathbf{u} - \mathbf{u}_\mathrm{l}) \quad \text{in } \mathbb{R}\ ,\\ \mathbf{u}(-\infty) = \mathbf{u}_\mathrm{l},\ \ \mathbf{u}(+\infty) = \mathbf{u}_\mathrm{r}\ , \end{cases} \tag{11.22}$$

when the wave speed s satisfies (11.14). If this system has a solution, then the shock $\{\mathbf{u}_\mathrm{l}, \mathbf{u}_\mathrm{r}, s\}$ is admissible. Such admissible shocks, however, need not satisfy the Lax-conditions. Examples, references and illuminating discussions are given in ISAACSON ET AL. [39]. In particular *transitional* shocks may occur satisfying

$$\lambda_k(\mathbf{u}_\mathrm{l}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{l}) \ , \tag{11.23a}$$

$$\lambda_k(\mathbf{u}_\mathrm{r}) < s < \lambda_{k+1}(\mathbf{u}_\mathrm{r}) \ . \tag{11.23b}$$

For $n = 2$ and D such that $\det(D) > 0$, this means that the travelling wave is represented by an orbit in $\mathbb{R}^2$ connecting two saddles. Such orbits are structurally unstable and may disappear under small perturbations of $\mathbf{u}_\mathrm{l}$ and $\mathbf{u}_\mathrm{r}$, and of D. In other words, keeping $\mathbf{u}_\mathrm{l}$ fixed and changing the viscosity matrix D, a connecting orbit can only be found (if any at all) by adjusting $\mathbf{u}_\mathrm{r}$. Consequently, the values of $\mathbf{u}$ at the shock depend on the local structure implied by the parabolic regularization. This behaviour is studied in detail by BRUINING & VAN DUIJN [12] for a specific problem related to oil recovery by steamdrive. This paper is included as Chapter 13. First we are going to discuss some elements of multi-phase flow in porous media.

12 Multi-phase flow in porous media

Oil occurs in the pores of reservoir rock. When the rock is permeable, the oil may move through the pores. For example the oil could be driven out of a reservoir by injecting water. We have a two-phase system if only water and oil occupy the pores. In addition, if gas is present, we have a three-phase system. The following concepts are important to understand the physics of multi-phase flow in porous media. For further details the reader is referred to, for example, BEAR [10], HELMIG [35] and AZIZ & SETTARI [8].

(*i*) *Porosity* (Φ) [$\mathrm{m}^3_{\mathrm{void}}/\mathrm{m}^3_{\mathrm{rock}}$] := fraction of 'voids' in the porous rock. Porosities range typically between 5-35%.

(*ii*) *Saturation* (S) [$\mathrm{m}^3_{\mathrm{fluid}}/\mathrm{m}^3_{\mathrm{void}}$] := fraction of the pore filled with a particular fluid.

(*iii*) *Connate water saturation* (S_{wc}) := maximum water saturation capillarily trapped. A typical porous medium has a connate water saturation $S_{\mathrm{wc}} = 0.2$ and an oil saturation $S_{\mathrm{o}} = 0.8$. Application of a pressure gradient only causes the oil to flow. The water is trapped.

(*iv*) *Residual oil saturation* (S_{or}) := maximum oil saturation capillarily trapped.

(*v*) *Residual gas saturation* (S_{gr}) := maximum gas saturation capillarily trapped.

(*vi*) *Capillary pressure* (P_{c}) [Pa] := average pressure difference between the fluid phases due to curved interfaces between them.

(*vii*) *Darcy's law* : fluids in porous media are driven by pressure gradients $(\mathrm{grad}\, P)$ and by gravitational forces. The latter are disregarded here for the sake of simplicity. The flow is given in terms of the volume flux or specific discharge $\mathbf{q}$ [$\mathrm{m}^3_{\mathrm{fluid}}/\mathrm{m}^2_{\mathrm{rock}} s$]. Analogous to laminar flow in interconnected tubes we have for one-phase flow

$$\mathbf{q} = -\frac{k}{\mu}\,\mathrm{grad}\, P\ , \tag{12.1}$$

where μ [Pa s] denotes the fluid viscosity and k [m^2] the rock permeability. Note that k is constant for homogeneous and isotropic rock.

12.1 Two-phase flow

Let water (saturation $S_\mathrm{w} \in [S_\mathrm{wc}, 1-S_\mathrm{or}]$) and oil (saturation $S_\mathrm{o} \in [S_\mathrm{or}, 1-S_\mathrm{wc}]$) occupy the pores of a homogeneous and isotropic porous medium. Then

$$S_\mathrm{w} + S_\mathrm{o} = 1 \; . \tag{12.2}$$

A modification of (12.1) is used in two-phase flow. If water flows through the medium with water and oil present in the pores, we have for the water discharge

$$\mathbf{q}_\mathrm{w} = -\frac{k_\mathrm{w}}{\mu_\mathrm{w}} k \operatorname{grad} P \; . \tag{12.3a}$$

Here $k_\mathrm{w} = k_\mathrm{w}(S_\mathrm{w})$ is called the relative permeability of water. Similarly,

$$\mathbf{q}_\mathrm{o} = -\frac{k_\mathrm{o}}{\mu_\mathrm{o}} k \operatorname{grad} P \; , \tag{12.3b}$$

where $k_\mathrm{o} = k_\mathrm{o}(S_\mathrm{o})$ is the relative permeability of oil.

There is little theoretical evidence for expressions (12.3). Under restrictive conditions, MIKELIC & PAOLI [52] were able to obtain analytical expressions using homogenisation techniques. In general, the relative permeabilities are obtained from experiments. They satisfy the following structural properties:

> $k_\mathrm{rw} : [S_\mathrm{wc}, 1-S_\mathrm{or}] \to [0,1]$, such that $k_\mathrm{rw}(S_\mathrm{wc}) = 0$, $k_\mathrm{w}(S_\mathrm{w}) > 0$ and strictly increasing for $S_\mathrm{wc} < S_\mathrm{w} < 1 - S_\mathrm{or}$,

and we expect $k_\mathrm{w}(1-S_\mathrm{or}) = 1$, reflecting one-phase flow;

> $k_\mathrm{ro} : [S_\mathrm{or}, 1 - S_\mathrm{wc}] \to [0,1]$, such that $k_\mathrm{ro}(S_\mathrm{or}) = 0$, $k_\mathrm{o}(S_\mathrm{o}) > 0$ and strictly increasing for $S_\mathrm{or} < S_\mathrm{o} < 1 - S_\mathrm{wc}$,

and again we expect $k_\mathrm{o}(1-S_\mathrm{wc}) = 1$.

However, there is a difference related to the wetting properties of the medium. When a porous medium is water-wet, water at low saturation tends to withdraw in the corners of the pores. The result is a slight reduction in the relative oil permeability: $k_\mathrm{o}(1-S_\mathrm{wc}) = k'' < 1$. Oil at low saturation forms bubbles in the middle of the pores. The result is a more significant reduction in the relative water permeability: $k_\mathrm{w}(1 - S_\mathrm{or}) = k'$ with $k' < k''$. Summarizing we expect the relative permeabilities to behave as in Figure 12.1.

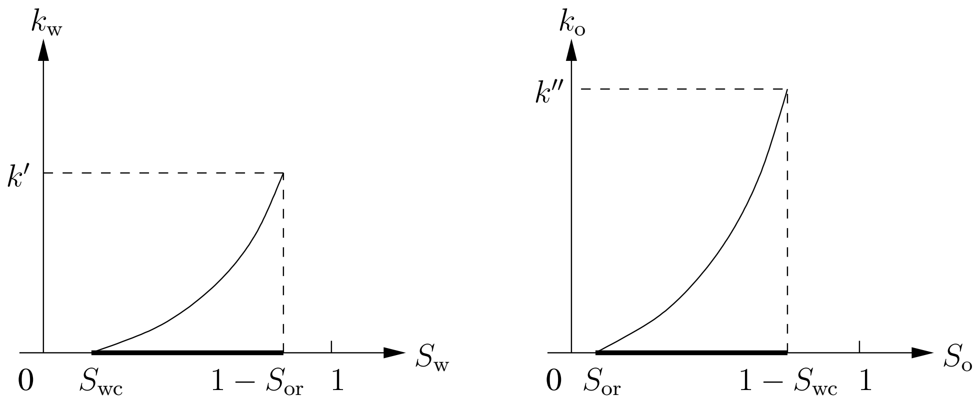


Figure 12.1. *Relative permeabilities*

For convenience we scale the saturations and set

$$S_\mathrm{wd} = \frac{S_\mathrm{w} - S_\mathrm{wc}}{1 - S_\mathrm{wc} - S_\mathrm{or}} \tag{12.4a}$$

and

$$S_\mathrm{od} = \frac{S_\mathrm{o} - S_\mathrm{or}}{1 - S_\mathrm{wc} - S_\mathrm{or}} \ . \tag{12.4b}$$

In the discussion below we assume that they have the same saturation dependence. In petroleum engineering terms we assume that they are of Corey-type, with

$$k_\mathrm{w} = k' S_\mathrm{wd}^2 \ , \qquad k_\mathrm{o} = k'' S_\mathrm{od}^2 \ , \tag{12.4c}$$

where kk' is the permeability of water at residual oil saturation and kk'' the permeability of oil at connate water saturation. Note that these relative permeabilities are not entirely consistent with the physical arguments presented above.

The conservation equation for a phase is obtained from its mass balance. As in open space we have for an arbitrary volume V, enclosed by the surface S,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho'_\alpha \mathrm{d}V + \oint_S \rho'_\alpha \mathbf{q}'_\alpha \cdot \mathbf{n} \, \mathrm{d}S = 0 \ . \tag{12.5}$$

The density ρ'_α of the fluid phase ($\alpha \in \{\mathrm{o}, \mathrm{w}\}$) is expressed in terms of the mass of fluid per unit rock volume. In the same way $\mathbf{q}'$ expresses the volume flux in terms of volume of fluid per unit cross-section of pores available to fluids of phase α.

It is more convenient to express the mass balance in terms of the fluid density ρ_α and the specific discharge $\mathbf{q}_\alpha$ They are related to ρ'_α and $\mathbf{q}'_\alpha$ by

$$\rho'_\alpha = \rho \Phi S_\alpha \tag{12.6a}$$

and

$$\mathbf{q}'_\alpha = \frac{\mathbf{q}_\alpha}{\Phi S_\alpha} \ . \tag{12.6b}$$

Hence we obtain for the mass balance equation in the porous medium

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \Phi S_\alpha \rho_\alpha \,\mathrm{d}V + \oint_S \rho_\alpha \mathbf{q}_\alpha \cdot \mathbf{n}\,\mathrm{d}S = 0\ . \tag{12.7}$$

When the variables appearing in (12.7) are sufficiently smooth we can apply the divergence theorem. This gives the mass balance equation in differential form

$$\frac{\partial}{\partial t}\left(\Phi S_\alpha \rho_\alpha\right) + \mathrm{div}\,\left(\rho_\alpha \mathbf{q}_\alpha\right) = 0\ . \tag{12.8}$$

If in addition the fluid phases are incompressible and the porosity is constant, we find

$$\Phi\frac{\partial S_\alpha}{\partial t} + \mathrm{div}\,\mathbf{q}_\alpha = 0\ . \tag{12.9}$$

In the discussion below we restrict ourselves to one dimensional in a horizontal layer of porous rock. Let us assume that the layer has constant (unit) thickness and that the rock is homogeneous and isotropic. Then $\mathbf{q}_\alpha = q_\alpha \mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector in the horizontal flow direction. As a result, equation (12.9) reduces to

$$\Phi\frac{\partial S_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x} = 0\ . \tag{12.10}$$

Summing these equations for $\alpha = \mathrm{w}, \mathrm{o}$ and using (12.2), gives for $q := q_\mathrm{w} + q_\mathrm{o}$

$$\frac{\partial q}{\partial x} = 0, \qquad \text{implying} \qquad q = q(t) \qquad \text{only.}$$

If fluid (water or a mixture of water and oil) is injected from the left at the constant rate $q_{\text{inj}} > 0$, we find

$$q_\mathrm{w} + q_\mathrm{o} = q_{\text{inj}}\ , \tag{12.11}$$

throughout the flow domain and for all t.

Next we use Darcy's law (12.3) to express q_w and q_o in terms of S_w. Here the capillary pressure enters. Since only two phases are involved we have

$$P_\mathrm{c} = P_\mathrm{o} - P_\mathrm{w} \tag{12.12}$$

as the averaged pressure difference between the phases in a small control volume. In petroleum engineering it is usually described by the Leverett model, LEVERETT [48], which gives

$$P_\mathrm{c} = P_\mathrm{c}(S_\mathrm{w}) = \sigma\sqrt{\frac{\Phi}{k}}J(S_\mathrm{w})\ , \tag{12.13}$$

where σ denotes the interfacial tension between the phases in the pores and J the Leverett function. Disregarding hysteretic effects, experiments show that

$$\begin{cases} J : (S_{\mathrm{wc}}, 1 - S_{\mathrm{or}}] \to [0, \infty) \\ \text{with} \\ J \text{ strictly decreasing and } J(S_{\mathrm{wc}}-) = +\infty\ . \end{cases} \tag{12.14}$$

Using (12.12) and (12.13) we write

$$q_{\mathrm{o}} = -\frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} k \frac{\partial P_{\mathrm{o}}}{\partial x} = -\frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} k \frac{\partial P_{\mathrm{c}}}{\partial x} - \frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} k \frac{\partial P_{\mathrm{w}}}{\partial x}$$
$$= -\frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} k \frac{\partial P_{\mathrm{c}}}{\partial x} + \frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} \frac{\mu_{\mathrm{w}}}{k_{\mathrm{w}}} q_{\mathrm{w}} .$$

Substitution into (12.11) gives

$$q_{\mathrm{w}} \left(1 + \frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} \frac{\mu_{\mathrm{w}}}{k_{\mathrm{w}}} \right) = q_{\mathrm{inj}} + \frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} k \frac{\partial P_{\mathrm{c}}}{\partial x}$$

or

$$q_{\mathrm{w}} = \frac{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}}}{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} + \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}} q_{\mathrm{inj}} + \frac{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}}{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} + \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}} k \frac{\partial P_{\mathrm{c}}}{\partial x} .$$

Hence, using (12.10) for the water phase, we find for S_{w} the nonlinear convection-diffusion equation

$$\Phi \frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial}{\partial x} \left\{ \frac{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}}}{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} + \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}} q_{\mathrm{inj}} + \frac{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}}{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} + \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}} k \frac{\partial P_{\mathrm{c}}}{\partial x} \right\} = 0 .$$

To put this equation in dimensionless form we introduce a reference length L, definition (12.4a) – which we denote again by S_{w} – and expressions (12.4c). Setting

$$x := \frac{x}{L} ,$$
$$t := \frac{q_{\mathrm{inj}}}{L\Phi(1 - S_{\mathrm{wc}} - S_{\mathrm{or}})} t ,$$
$$M := \frac{\mu_{\mathrm{o}}}{\mu_{\mathrm{w}}} \frac{k'}{k''} \qquad \text{(viscosity ratio)} , \tag{12.15a}$$
$$N_{\mathrm{c}} := \frac{k'' \sigma \sqrt{k\Phi}}{q_{\mathrm{inj}}\, \mu_{\mathrm{o}} L} \qquad \text{(capillary number)} , \tag{12.15b}$$

yields the equation

$$\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial}{\partial x} \left\{ F_{\mathrm{w}}(S_{\mathrm{w}}) - N_{\mathrm{c}} D(S_{\mathrm{w}}) \frac{\partial S_{\mathrm{w}}}{\partial x} \right\} = 0 , \tag{12.16}$$

where

$$F_{\mathrm{w}}(S_{\mathrm{w}}) = \frac{M S_{\mathrm{w}}^2}{S_{\mathrm{o}}^2 + M S_{\mathrm{w}}^2} = \frac{M S_{\mathrm{w}}^2}{(1 - S_{\mathrm{w}})^2 + M S_{\mathrm{w}}^2}$$

is called the fractional flow function and where

$$D(S_{\mathrm{w}}) = -F_{\mathrm{w}}(S_{\mathrm{w}})(1 - S_{\mathrm{w}})^2 \frac{\mathrm{d}J}{\mathrm{d}S_{\mathrm{w}}}$$

denotes the capillary induced diffusion. The Leverett function is expressed here in terms of the scaled water saturation. In petroluem engineering this equation is often considered in the limit of vanishing capillary forces (i.e. $N_c \downarrow 0$), which yields the *Buckley–Leverett equation*

$$\frac{\partial S_w}{\partial t} + \frac{\partial F_w(S_w)}{\partial x} = 0 \,. \tag{12.17}$$

One easily verifies

$$F_w'(S_w) > 0 \quad \text{for} \quad 0 < S_w < 1 \,, \;\; F_w'(0) = F_w'(1) = 0 \,;$$

$$F_w''(S_w) \begin{cases} > 0 & \text{for } 0 < S_w < \widetilde{S} \,, \\ < 0 & \text{for } \widetilde{S} < S_w < 1 \,, \end{cases} \quad \text{where } \widetilde{S} \in (0,1) \text{ depends on } M \,;$$

$$F_w'(S_w) = \frac{F_w(S_w)}{S_w} \quad \Longleftrightarrow \quad S_w = \sqrt{\frac{1}{1+M}} \,.$$

Hence the fractional flow function satisfies all the properties required in Section 6. Consequently, the solutions of the Riemann problems considered there are directly applicable to the Buckley–Leverett equation (12.17).

12.2 Some remarks on degenerate diffusion

The diffusivity in (12.16) depends on the water saturation and satisfies

$$\begin{cases} D(S_w) > 0 & \text{for } 0 < S_w < 1 \,, \\ D(0) = D(1) = 0 \,, \end{cases} \tag{12.18a}$$

provided $J' < 0$ on $(0,1]$ and $-S_w^2 \dfrac{\mathrm{d}J(S_w)}{\mathrm{d}S_w} \downarrow 0$ as $S_w \downarrow 0$. In fact, since k_w and k_o are given by (12.4c), we have

$$D(S_w) = \mathcal{O}\big((1-S_w)^2\big) \quad \text{as } S_w \uparrow 1 \tag{12.18b}$$

and let us suppose that

$$D(S_w) = \mathcal{O}\big(S_w^p\big) \quad (p > 0) \quad \text{as } S_w \downarrow 0 \,. \tag{12.18c}$$

Equations of the form (12.16), with D satisfying (12.18), are called degenerate parabolic. The degeneration occurs at $S_w = 0$ and $S_w = 1$ where the diffusivity vanishes. Such equations received much attention in the mathematics literature over the past decades. The first papers go back as early as the 1950's, see for instance OLEINIK ET AL. [56], and were mainly concerned with the so-called *porous media equation*:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2}{\partial x^2}(u^m) & (m > 1) \,, \\ u \geqslant 0 \,. \end{cases} \tag{12.19}$$

This equation describes the flow of a gas (u denotes density) in a porous medium; details are given in MUSKAT [55]. The diffusivity in (12.19), i.e. $D(u) = mu^{m-1}$, vanishes as $u \downarrow 0$. This property implies the existence of *interfaces* or *free boundaries* in the $x-t$ plane, separating the regions where $u>0$ and where $u=0$. The following explicit solution, found independently by BARENBLATT [9] and PATTLE [58], shows this behaviour. Consider (12.19) for $x \in \mathbb{R}$ and $t>0$, subject to

$$u(\cdot,0) = M\delta(\cdot) \qquad (M>0) \quad \text{ on } \mathbb{R}\,,$$

where δ denotes the Dirac distribution at the origin. The unique solution is given by

$$u(x,t) = t^{-\alpha}\Big\{\big[A - Bx^2t^{-2\alpha}\big]_+\Big\}^{\frac{1}{m-1}}\,, \tag{12.20}$$

where $[\cdot]_+ = \max\{\cdot,0\}$, $\alpha = \dfrac{1}{m+1}$, $B = \dfrac{m-1}{2m(m+1)}$ and where A is a positive constant depending on m and M. Setting

$$r(t) := \sqrt{\frac{A}{B}}\,t^{\frac{1}{m+1}} \qquad \text{for } t \geqslant 0\,,$$

we observe that the curves

$$\big\{(x,t) : t \geqslant 0,\;\; x = \pm r(t)\big\}$$

form the boundaries of the expanding support of u such that $u(x,t)>0$ for $|x|<r(t)$, while $u(x,t)=0$ for all $|x| \geqslant r(t)$. Thus the material, initially concentrated at $x=0$, spreads with finite speed of propagation in space, see Figure 12.2.

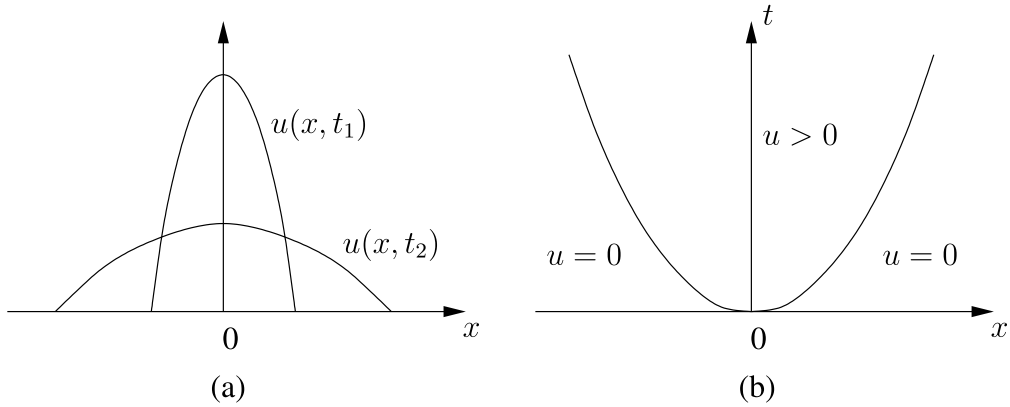


Figure 12.2. *Sketch of the Barenblatt–Pattle solution:* (a) *profiles for* $0<t_1<t_2$*;* (b) *free boundaries in the* $x-t$ *plane*

The free boundaries occur for any $m>1$. When $m=1$, equation (12.19) reduces to the linear heat equation. Then

$$u(x,t) = \frac{M}{\sqrt{4\pi t}}\exp\left\{-\frac{x^2}{4t}\right\}\,,$$

which implies that $u(\cdot,t)$ for any $t>0$. Now the material spreads with infinite speed of propagation.

Remark 12.1. *The Barenblatt–Pattle solution* (12.20) *is obtained by the same method as used in section 2 of Chapter 1. One sets* $u(x,t) = t^{-\alpha} f(\eta)$ *with* $\eta = xt^{-\beta}$, *and requires* $\int_{-\infty}^{+\infty} u(x,t)\mathrm{d}x = M$. *This gives* $\alpha = \beta$. *Substituting the self-similar form into* (12.19) *implies* $\alpha = \dfrac{1}{1+m}$ *and results in an ordinary differential equation for* f.

For arbitrary non-negative initial data with bounded support, a complete theory has been developed for equation (12.19). This involves optimal regularity results, properties and smoothness of the free boundaries and the large time behaviour – stabilization – of solutions. Much of these results have been generalized to higher space dimensions ($n > 1$) and to more general nonlinearities. A survey is given by ARONSON [4].

Next we return to the water-oil case. Suppose no fluid is injected from the left into the horizontal column (i.e. $q_{\text{inj}} = 0$) and that only redistribution of the fluids due to capillary forces is considered. Slightly redefining N_{c} and absorbing it in the dimensionless time, gives for S_w the equation

$$\frac{\partial S_{\text{w}}}{\partial t} = \frac{\partial}{\partial x}\big(D(S_{\text{w}})\frac{\partial S_{\text{w}}}{\partial x}\big) , \tag{12.21}$$

with D satisfying (12.18a). Solving this equation for $x \in \mathbb{R}$ and $t > 0$, subject to the initial distribution

$$S_{\text{w}}(\cdot, 0) = S_{\text{in}}(x) \qquad \text{on } \mathbb{R} ,$$

where S_{in} distinguishes the regions

$$S_{\text{in}}(x) = \begin{cases} 1 & \text{as } -\infty < x \leqslant a & \text{(water region)} \\ \in (0,1) & \text{as } a < x < b & \text{(both fluids present)} \\ 0 & \text{as } b \leqslant x < \infty & \text{(oil region)} , \end{cases}$$

we expect two free boundaries to arise. One starting at $(x,t) = (a,0)$ between the regions where $S_{\text{w}} = 1$ and $S_{\text{w}} < 1$, and one starting at $(x,t) = (b,0)$ between the regions where $S_{\text{w}} > 0$ and $S_{\text{w}} = 0$.

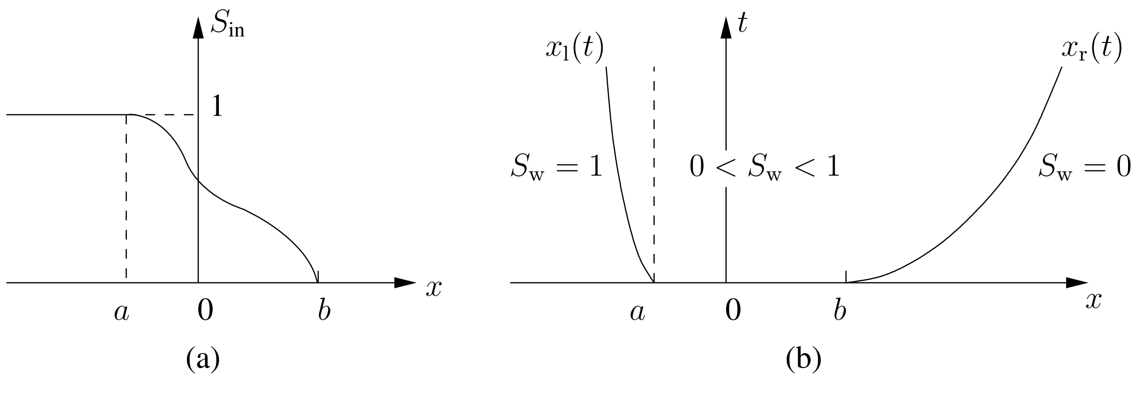


Figure 12.3. *Free boundaries in two-phase flow:* (a) *initial distribution* S_{in}; (b) *free boundaries in* $x - t$ *plane. Because of the strong capillary effect near* $S_{\text{w}} = 0$, *the right free boundary moves faster than the left free boundary*

These qualitative impressions have been made rigorously by VAN DUIJN & ZHANG [74] and VAN DUIJN & FLORIS [73]. A particular case arises when considering the Riemann problem for (12.21), where

$$S_{\text{in}}(x) = \begin{cases} 1 & \text{as } x < 0 \ , \\ 0 & \text{as } x > 0 \ . \end{cases} \tag{12.22}$$

As in the case of rarefaction waves for first order hyperbolic equations, the initial value problem (12.21), (12.22) can be reduced to a boundary value problem for the self-similar solution

$$S(x,t) = s(\eta) \ , \qquad \text{with } \eta = x/\sqrt{t} \ .$$

For s results

$$\text{(RD)} \begin{cases} \dfrac{1}{2}\eta\dfrac{\mathrm{d}s}{\mathrm{d}\eta} + \dfrac{\mathrm{d}}{\mathrm{d}\eta}\left(D(s)\dfrac{\mathrm{d}s}{\mathrm{d}\eta}\right) = 0 \quad \text{in } \mathbb{R} \ , \\ s(-\infty) = 1 \ , \ \ s(+\infty) = 0 \ . \end{cases}$$

This problem has a unique continuous solution $s:\mathbb{R}\to[0,1]$ and there exist $a, b\in\mathbb{R}$, $-\infty < a < 0 < b < \infty$, such that

$$s(\eta) = \begin{cases} 1 & \text{as } \eta \leqslant a \ , \\ \text{strictly decreasing} & \text{as } a < \eta < b \ , \\ 0 & \text{as } \eta \geqslant b \ , \end{cases}$$

see VAN DUIJN & PELETIER [75]. The numbers a and b imply the free boundaries

$$x_{\text{l}}(t) = a\sqrt{t} < 0 \quad \text{and} \quad x_{\text{r}}(t) = b\sqrt{t} > 0 \ .$$

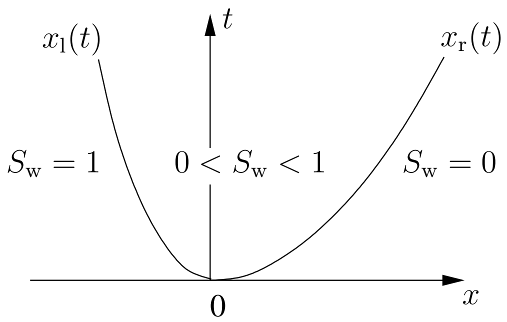


Figure 12.4. *Free boundaries* $\left(x_{\text{l}}(t)=a\sqrt{t},\ x_{\text{r}}(t)=b\sqrt{t}\right)$ *in the Riemann problem* (12.21), (12.22)

The appearance of the free boundaries critically depends on the behaviour of $D(S_{\text{w}})$ near $S_{\text{w}} = 0$, $S_{\text{w}} = 1$. The precise conditions are (ATKINSON & PELETIER [6, 7], VAN DUIJN & FLORIS [73]): for some $\delta > 0$

$$\int_0^{\delta} \frac{D(s)}{s}\,\mathrm{d}s < \infty \quad \Longleftrightarrow \quad b < \infty$$

and

$$\int_{1-\delta}^{1} \frac{D(s)}{1-s} \, \mathrm{d}s < \infty \quad \Longleftrightarrow \quad a > -\infty \, .$$

With D satisfying (12.18b,c) we have $-\infty < a < 0 < b < \infty$, as shown in Figure 12.4. If $D(S_\mathrm{w}) = D_0$ (constant > 0), then $a = -\infty$ and $b = +\infty$ and no free boundaries are present. The solution is found by direct integration and reads

$$s(\eta) = \frac{1}{2} \operatorname{erfc}\left(\frac{\eta}{2\sqrt{D_0}}\right) \qquad \text{for } \eta \in \mathbb{R} \, ,$$

where

$$\operatorname{erfc}(p) = \frac{2}{\sqrt{\pi}} \int_p^\infty \exp\left\{-z^2\right\} \mathrm{d}z \, .$$

Now $s(\eta) \in (0,1)$ for all $\eta \in \mathbb{R}$, implying $S_\mathrm{w}(\cdot, t) \in (0,1)$ for any $t > 0$. Thus water and oil are present everywhere, for any $t > 0$. This means infinite speed of propagation.

Free boundaries, if they exist, satisfy equations (so-called *free boundary equations*) that are based on the local fluid balance. With reference to Figure 12.3 we have

$$\frac{\mathrm{d}x_\mathrm{l}}{\mathrm{d}t}(t) = \lim_{x \downarrow x_\mathrm{l}(t)} \frac{D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{1 - S_\mathrm{w}}(x,t) \tag{12.23a}$$

and

$$\frac{\mathrm{d}x_\mathrm{r}}{\mathrm{d}t}(t) = \lim_{x \uparrow x_\mathrm{r}(t)} \frac{D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{S_\mathrm{w}}(x,t) \, , \tag{12.23b}$$

expressing that the speed of a free boundary is given by the speed of the fluid particles (oil for x_l and water for x_r). Equations (12.23) determine the optimal regularity for solutions. They are smooth functions of x and t whenever $S_\mathrm{w}(x,t) \in (0,1)$ (i.e. between the free boundaries), but

$$\lim_{S_\mathrm{w} \uparrow 1} \frac{D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{1 - S_\mathrm{w}} = \mathcal{O}(1) \quad \text{and} \quad \lim_{S_\mathrm{w} \downarrow 0} \frac{D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{S_\mathrm{w}} = \mathcal{O}(1) \, .$$

Using (12.18b,c) this means

$$1 - S_\mathrm{w}(x,t) = \mathcal{O}\big(\sqrt{x - x_\mathrm{l}(t)}\big) \quad \text{as } x \downarrow x_\mathrm{l}(t) \tag{12.24a}$$

and

$$S_\mathrm{w}(x,t) = \mathcal{O}\big(\sqrt[p]{x_\mathrm{r}(t) - x}\big) \quad \text{as } x \uparrow x_\mathrm{r}(t) \, . \tag{12.24b}$$

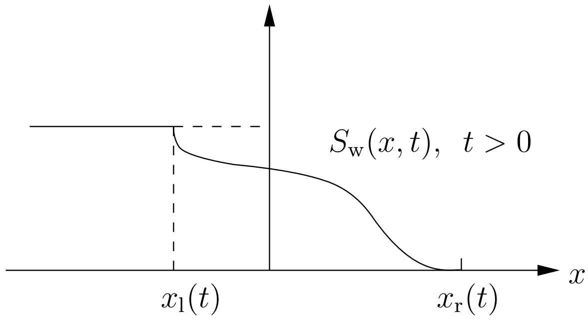


Figure 12.5. *Qualitative behaviour of water saturation near the free boundaries. Here* $p \in (0,1)$. *The difference is caused by the behaviour of the capillary pressure*

Rigorous results concerning properties and regularity of free boundaries induced by degenerate diffusion were first obtained by ARONSON [5] for the porous media equation (12.19). Extensions to doubly degenerate equations were given by BERTSCH ET AL. [11] and VAN DUIJN & FLORIS [73]. When applied to the self-similar solution of (RD), equations (12.23) reduce to

$$\frac{1}{2}a = \lim_{\eta \downarrow a} \frac{D(s)\dfrac{\mathrm{d}s}{\mathrm{d}\eta}}{1-s} \quad \text{and} \quad \frac{1}{2}b = -\lim_{\eta \uparrow b} \frac{D(s)\dfrac{\mathrm{d}s}{\mathrm{d}\eta}}{s} \ .$$

This behaviour is demonstrated by VAN DUIJN & PELETIER [75].

Including nonlinear convection as in equation (12.16) obviously changes the movement of the free boundaries. However, with k_w and k_o satisfying (12.4c), their equations remain the same. We now have, again with reference to Figure 12.3,

$$\begin{aligned} \frac{\mathrm{d}x_\mathrm{l}}{dt}(t) &= \lim_{x \downarrow x_\mathrm{l}(t)} \frac{F_\mathrm{w}(1) - F_\mathrm{w}(S_\mathrm{w}) + N_\mathrm{c} D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{1 - S_\mathrm{w}}(x,t) \\ &= F_\mathrm{w}'(1) + N_\mathrm{c} \lim_{x \downarrow x_\mathrm{l}(t)} \frac{D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{1 - S_\mathrm{w}}(x,t) \end{aligned} \tag{12.25a}$$

and

$$\begin{aligned} \frac{\mathrm{d}x_\mathrm{r}}{dt}(t) &= \lim_{x \uparrow x_\mathrm{r}(t)} \frac{F_\mathrm{w}(S_\mathrm{w}) - N_\mathrm{c} D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{S_\mathrm{w}}(x,t) \\ &= F_\mathrm{w}'(0) - N_\mathrm{c} \lim_{x \uparrow x_\mathrm{r}(t)} \frac{D(S_\mathrm{w})\dfrac{\partial S_\mathrm{w}}{\partial x}}{S_\mathrm{w}}(x,t) \ , \end{aligned} \tag{12.25b}$$

but $F_\mathrm{w}'(1) = F_\mathrm{w}'(0) = 0$. Thus even in the presence of the convective term we have $\dfrac{\mathrm{d}x_\mathrm{l}}{\mathrm{d}t} \leqslant 0$ for all $t > 0$ and for all $N_\mathrm{c} > 0$, implying that the left free boundary moves against the direction of the flow due to the capillary forces. This is to be expected from the hyperbolic limit, since the solution of the Riemann problem (12.17), (12.22) starts with a rarefaction to the right of $x = 0$ with characteristic speed $F'(1) = 0$.

If we would change the fractional flow function such that $F'(1) > 0$, then the convective speed is strictly positive at $S_w = 1$ and will eventually overcome the capillary induced movement. Consequently, the water completely sweeps the accessible oil from the porous rock. This follows trivially if we set $F_w(S_w) = S_w$ in (12.16), while keeping $D(S_w)$ as in (12.18), and consider the solution satisfying (12.22) at $t = 0$. Setting

$$S_w(x,t) = s(\eta) , \quad \text{now with} \quad \eta = \frac{x - t}{N_c \sqrt{t}} ,$$

gives for s again (RD). Consequently,

$$x_l(t) = t + a\sqrt{N_c t} \qquad (a < 0)$$

and

$$x_r(t) = t + b\sqrt{N_c t} \qquad (b > 0)$$

for all $t \geqslant 0$.

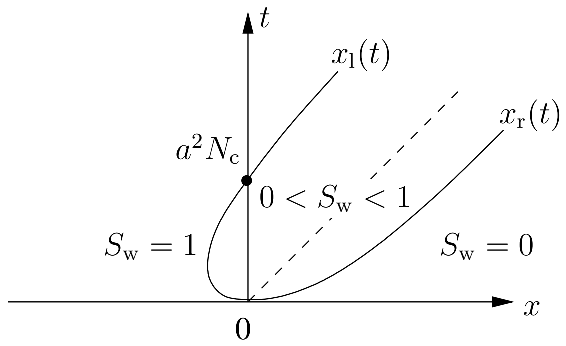


Figure 12.6. *Behaviour of the free boundaries when $F_w(S_w) = S_w$, with* (12.22) *at* $t = 0$

From (12.25) we note that the qualitative behaviour of S_w near the free boundaries remains the same in the presence of F_w (as long as $F_w \in C^1([0,1])$). Again (12.24) is satisfied.

One dimensional convection-degenerate diffusion equations received much attention in studies by Gilding, see [27, 28] for an overview. His work includes necessary and sufficient conditions for the occurence of free boundaries and a detailed analysis of their qualitative properties.

Remark 12.2. *One dimensional two-phase flow problems are special because* (12.11) *is satisfied with q_{inj} given. It allows for a reduction to a single equation for one of the saturations. For flows in $\mathbb{R}^n$ (n=2,3) one still has a transport equation for the (water) saturation, but this equation contains the unknown discharge* $\mathbf{q}$ *(properly scaled). A derivation similar to the one presented here yields*

$$\frac{\partial S_w}{\partial t} + \operatorname{div}\big(F_w(S_w)\mathbf{q} - N_c D(S_w) \operatorname{grad} S_w\big) = 0 , \tag{12.26}$$

with F_w and D as in (12.16)*, and*

$$\operatorname{div} \mathbf{q} = 0 . \tag{12.27}$$

There are various way to close this system. Assuming water to be present everywhere in the flow domain, one can combine (12.27)*, Darcy's law for water and oil, and the capillary pressure to obtain an equation for the water pressure from which* $\mathbf{q}$ *can be recovered. This follows from (in non-dimensionless variables)*

$$\begin{aligned} \mathbf{q} = \mathbf{q}_\mathrm{o} + \mathbf{q}_\mathrm{w} &= -\frac{kk_\mathrm{o}}{\mu_\mathrm{o}} \operatorname{grad} P_\mathrm{o} - \frac{kk_\mathrm{w}}{\mu_\mathrm{w}} \operatorname{grad} P_\mathrm{w} \\ &= -\frac{kk_\mathrm{o}}{\mu_\mathrm{o}} \operatorname{grad} P_\mathrm{c}(S_\mathrm{w}) - \left(\frac{kk_\mathrm{w}}{\mu_\mathrm{w}} + \frac{kk_\mathrm{o}}{\mu_\mathrm{o}}\right) \operatorname{grad} P_\mathrm{w} \ , \end{aligned}$$

giving

$$\operatorname{div}\left(\left[\frac{kk_\mathrm{w}}{\mu_\mathrm{w}} + \frac{kk_\mathrm{o}}{\mu_\mathrm{o}}\right] \operatorname{grad} P_\mathrm{w}\right) = -\operatorname{div}\left(\frac{kk_\mathrm{o}}{\mu_\mathrm{o}} \operatorname{grad} P_\mathrm{c}(S_\mathrm{w})\right) \ .$$

After scaling this equation reads

$$\operatorname{div}\left((MS_\mathrm{w}^2 + (1-S_\mathrm{w})^2) \operatorname{grad} p_\mathrm{w}\right) = -\operatorname{div}\left((1-S_\mathrm{w})^2 \operatorname{grad} J(S_\mathrm{w})\right) \ ,$$

where $p_\mathrm{w} := \dfrac{P_\mathrm{w}}{\sigma\sqrt{\frac{\Phi}{k}}}$*. Hence,* p_w*, and thereby* $\mathbf{q}$*, depends in a nonlocal way on* S_w*. This makes the analysis of* (12.26) *extremely hard and a qualitative study (as in the one-dimensional case) virtually impossible. Instead of using the water pressure* p_w*,* CHAVENT & JAFFRÉ [14] *introduced a global pressure for which an equation can be derived in a similar way (but without assuming* S_w *everywhere in the flow domain). Either way, one has to deal with an elliptic-parabolic system for pressure and saturation. Such systems were studied by* ALT & DIBENIDETTO [3]*, who were the first to obtain existence and regularity results. The hyperbolic limit for such systems is far from understood. When* $N_\mathrm{c} \downarrow 0$*, small scale fingering may occur due to the viscosity difference of water and oil. This raises fundamental questions concerning the modelling of multi-dimensional flows in absence of capillarity (see* OTTO [57]*).*

12.3 Three-phase flow

As we shall see, an enormous complexity enters the analysis if gas as a third phase is present in the reservoir. Let us extend the two-phase description by introducing a gas saturation S_g such that

$$S_\mathrm{w} + S_\mathrm{o} + S_\mathrm{g} = 1 \ , \tag{12.28}$$

where

$$\begin{aligned} S_\mathrm{wc} &\leqslant S_\mathrm{w} \leqslant 1 - S_\mathrm{or} - S_\mathrm{gr} \ , \\ S_\mathrm{or} &\leqslant S_\mathrm{o} \leqslant 1 - S_\mathrm{wc} - S_\mathrm{gr} \ , \\ S_\mathrm{gr} &\leqslant S_\mathrm{g} \leqslant 1 - S_\mathrm{wc} - S_\mathrm{or} \ . \end{aligned}$$

Similar to (12.4) we consider scaled saturation $S_\mathrm{id} \in [0,1]$ $(\mathrm{i}=\mathrm{w},\mathrm{o},\mathrm{g})$ and, assuming for gas a Darcy law as well, we introduce the additional Corey relative permeability $k_\mathrm{g} = k''' S_\mathrm{gd}^2$. Then

$$\mathbf{q}_\mathrm{g} = -\frac{k_\mathrm{g}}{\mu_\mathrm{g}} k \operatorname{grad} P_\mathrm{g} \ .$$

To obtain the hyperbolic setting we disregard the capillary forces between the phases in the pores. This implies

$$P_{\mathrm{w}} = P_{\mathrm{o}} = P_{\mathrm{g}} \; . \tag{12.29}$$

Considering again horizontal flow in a homogeneous and isotropic porous layer of constant thickness, we now find

$$q_{\mathrm{w}} + q_{\mathrm{o}} + q_{\mathrm{g}} = q_{\mathrm{inj}} > 0 \; .$$

Using (12.29) and Darcy's law for the phases, we also have

$$q_{\mathrm{o}} = \frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}}\frac{\mu_{\mathrm{w}}}{k_{\mathrm{w}}} q_{\mathrm{w}} \quad \text{and} \quad q_{\mathrm{g}} = \frac{k_{\mathrm{g}}}{\mu_{\mathrm{g}}}\frac{\mu_{\mathrm{w}}}{k_{\mathrm{w}}} q_{\mathrm{w}} \; ,$$

giving

$$q_{\mathrm{w}} \left(\frac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} + \frac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} + \frac{k_{\mathrm{g}}}{\mu_{\mathrm{g}}} \right) = \frac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} q_{\mathrm{inj}} \; .$$

Hence for the water phase results

$$\Phi \frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}}}{\dfrac{k_{\mathrm{w}}}{\mu_{\mathrm{w}}} + \dfrac{k_{\mathrm{o}}}{\mu_{\mathrm{o}}} + \dfrac{k_{\mathrm{g}}}{\mu_{\mathrm{g}}}} q_{\mathrm{inj}} \right) = 0 \; .$$

In terms of the scaled saturations (dropping the index d) and the dimensionless variables

$$x := \frac{x}{L} \; , \quad t := \frac{q_{\mathrm{inj}}}{\Phi L (1 - S_{\mathrm{wc}} - S_{\mathrm{or}} - S_{\mathrm{gr}})} t \; ,$$

the water equation reads

$$\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial F_{\mathrm{w}}}{\partial x} = 0 \; ,$$

where

$$F_{\mathrm{w}} = F_{\mathrm{w}}(S_{\mathrm{w}}, S_{\mathrm{o}}, S_{\mathrm{g}}) = \frac{M_{\mathrm{ow}} S_{\mathrm{w}}^2}{M_{\mathrm{ow}} S_{\mathrm{w}}^2 + S_{\mathrm{o}}^2 + M_{\mathrm{og}} S_{\mathrm{g}}^2} \; ,$$

with

$$M_{\mathrm{ow}} = \frac{\mu_{\mathrm{o}}}{\mu_{\mathrm{w}}} \frac{k'}{k''} \quad \text{(oil-water viscosity ratio)} \; ,$$
$$M_{\mathrm{og}} = \frac{\mu_{\mathrm{o}}}{\mu_{\mathrm{g}}} \frac{k'''}{k''} \quad \text{(oil-gas viscosity ratio)} \; .$$

Concerning these viscosity ratios one typically has (see Table I in Chapter 13)

$$M_{\mathrm{og}} \gg 1 \quad \text{and} \quad M_{\mathrm{ow}} > 1 \; .$$

Similar equations are found for S_{o} and S_{g}. Since (12.28) holds it suffices to consider the equations for two phases only. Let us eliminate S_{o} from the equations and set

$$u := S_{\mathrm{w}} \ , \ \ v := S_{\mathrm{g}} \ ; \ \ a = M_{\mathrm{ow}} \ , \ \ b = M_{\mathrm{og}} \ .$$

Then for $\mathbf{u} = (u, v)^T$ results the system

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} = \mathbf{0} \ , \tag{12.30}$$

when $\mathbf{F} = (F_u, F_v)^T$ is given by

$$F_u(u,v) = \frac{au^2}{au^2 + (1-u-v)^2 + bv^2} \ , \tag{12.31a}$$

$$F_v(u,v) = \frac{bv^2}{au^2 + (1-u-v)^2 + bv^2} \ . \tag{12.31b}$$

Below we discuss some aspects of the Riemann problem involving this system: i.e. we look for solutions of (12.30) in $Q = \mathbb{R} \times \mathbb{R}^+$ subject to

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_{\mathrm{l}} & \text{for } x < 0 \ , \\ \mathbf{u}_{\mathrm{r}} & \text{for } x > 0 \ . \end{cases}$$

Since $u + v = 1 - S_{\mathrm{o}} \leqslant 1$, the solution $\mathbf{u}$ and the initial states $\mathbf{u}_{\mathrm{l}}$, $\mathbf{u}_{\mathrm{r}}$ are confined to the closed triangle

$$\mathcal{D} := \{(u,v) : u \geqslant 0, \ \ v \geqslant 0 \ \ \text{and} \ \ u + v \leqslant 1\} \ .$$

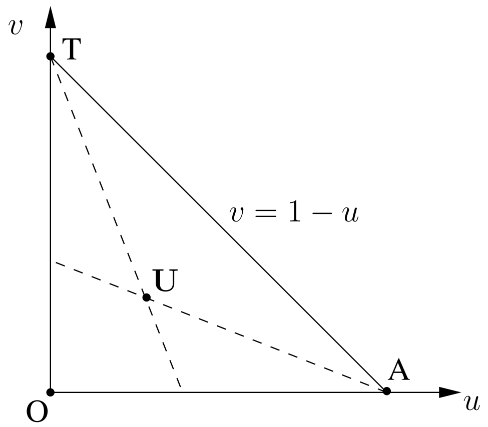


Figure 12.7. *Solution range $\mathcal{D}$ for three phase flow*

We first need to verify the hyperbolicity of (12.30) in $\mathcal{D}$. In general this depends critically on the choice of relative permeabilities. HOLDEN [36] and GUZMÁN & FAYERS [33, 32] studied this aspect in detail. Their work includes the so-called Stone permeabilities where $k_{\mathrm{w}} = k_{\mathrm{w}}(S_{\mathrm{w}})$, $k_{\mathrm{g}} = k_{\mathrm{g}}(S_{\mathrm{g}})$ and $k_{\mathrm{o}} = k_{\mathrm{o}}(S_{\mathrm{w}}, S_{\mathrm{g}})$. Here we restrict ourselves to the quadratic Corey expressions.

A point $\mathbf{U} \in \mathcal{D}$ is called an *umbilic point* if $\lambda_1(\mathbf{U}) = \lambda_2(\mathbf{U}) \in \mathbb{R}$. A region $R \subset \mathcal{D}$ is called an *elliptic region* if the eigenvalues $\lambda_1(\mathbf{U})$, $\lambda_2(\mathbf{U})$ form a complex conjugate pair for every $\mathbf{U} \in R$. We will show that (12.30) is strictly hyperbolic in $\mathcal{D}$, except at the vertices O, A and T, where $\lambda_1 = \lambda_2 = 0$, and at a unique interior point, the umbilic point $\mathbf{U} = (\frac{b}{a+b+ab}, \frac{a}{a+b+ab})^T$.

Introducing the notation $F_{ij} = \dfrac{\partial F_i}{\partial j}$ for $i, j = u, v$ we have

$$\lambda_{1,2} = \frac{1}{2}(F_{uu} + F_{vv}) \pm \frac{1}{2}\sqrt{d(\mathbf{u})}\,,$$

where

$$d(\mathbf{u}) = (F_{uu} - F_{vv})^2 + 4F_{uv}F_{vu}\,.$$

Differentiation gives

$$\begin{aligned}
F_{uu} &= \frac{1}{N^2}\left[2au\{(1-u-v)^2 + bv^2\} + 2au^2(1-u-v)\right]\,,\\
F_{vv} &= \frac{1}{N^2}\left[2bv\{(1-u-v)^2 + au^2\} + 2bv^2(1-u-v)\right]\,,\\
F_{uv} &= \frac{1}{N^2}2au^2\{(1-u-v) - bv\}\,,\\
F_{vu} &= \frac{1}{N^2}2bv^2\{(1-u-v) - au\}\,,
\end{aligned}$$

where $N = au^2 + (1-u-v)^2 + bv^2$. Along OT we have

$$\lambda_1 = 0\,,\quad \lambda_2 = \frac{2bv(1-v)}{\left\{(1-v)^2 + bv^2\right\}^2}\,;$$

along OA

$$\lambda_1 = 0\,,\quad \lambda_2 = \frac{2au(1-u)}{\left\{au^2 + (1-u)^2\right\}^2}\,;$$

and along AT

$$\lambda_1 = 0\,,\quad \lambda_2 = \frac{2abuv}{\left\{au^2 + bv^2\right\}^2}\,.$$

Hence $\lambda_1 = \lambda_2 = 0$ at O, A and T. Next we observe that

$$\begin{aligned}
F_{uv} > (<)\, 0 &\quad \text{if and only if} \quad v < (>)\, \frac{1-u}{1+b}\,,\\
F_{vu} > (<)\, 0 &\quad \text{if and only if} \quad u < (>)\, \frac{1-v}{1+a}\,.
\end{aligned}$$

Hence $P := F_{uv}F_{vu} < 0$ in the shaded regions in Figure 12.8, $P = 0$ along their boundaries and in particular at the point $\mathbf{U} = (\frac{b}{a+b+ab}, \frac{a}{a+b+ab})^T$, and $P > 0$ elsewhere.

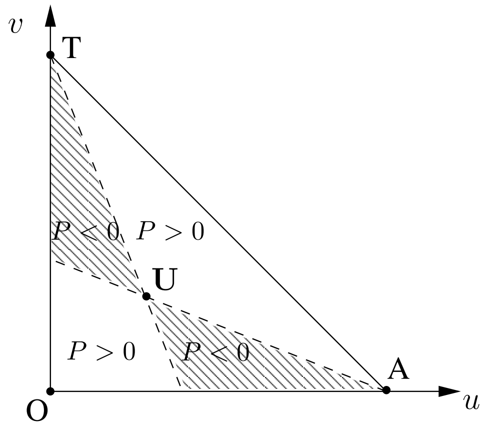


Figure 12.8. *Sign of the product* $P = F_{uv}F_{vu}$ *in* $\mathcal{D}$

Clearly (12.30) is strictly hyperbolic whenever $P > 0$. By direct substitution we find $F_{uu} = F_{vv}$ at $\mathbf{U}$. Thus $\mathbf{U}$ is indeed an interior umbilic point. To exclude other umbilic points or elliptic regions we need to show that $d > 0$ inside the shaded regions where $P < 0$ and along the boundaries $v = \dfrac{1-u}{1+b}$ and $u = \dfrac{1-v}{1+a}$, except at $\mathbf{U}$ where they intersect. For this purpose we write

$$\begin{aligned} N^2(F_{uu} - F_{vv}) &= 2au\big((1-u-v)^2 + bv^2\big) + 2au^2(1-u-v) \\ &\qquad - 2bv\big((1-u-v)^2 + au^2\big) - 2bv^2(1-u-v) \\ &= \big((1-u-v)^2 + bv^2\big)\big(2au - 2(1-u-v)\big) \\ &\qquad - \big((1-u-v)^2 + au^2\big)\big(2bv - 2(1-u-v)\big) , \end{aligned}$$

which gives

$$\begin{aligned} N^4(F_{uu} - F_{vv})^2 &= 4\big((1-u-v)^2 + bv^2\big)^2\big(au - (1-u-v)\big)^2 \\ &\quad - 8\big((1-u-v)^2 + bv^2\big)\big((1-u-v)^2 + au^2\big)\big(au - (1-u-v)\big)\big(bv - (1-u-v)\big) \\ &\quad + 4\big((1-u-v)^2 + au^2\big)^2\big(bv - (1-u-v)\big)^2 . \end{aligned}$$

Let $\mathbf{u} \neq \mathbf{U}$ be a point where $P \leqslant 0$. Then we estimate

$$\begin{aligned} \frac{1}{4}N^4(F_{uu} - F_{vv})^2 &> \big(bv^2\big)^2\big(au - (1-u-v)\big)^2 \\ &\quad - 2\big(bv^2\big)\big(au^2\big)\big(au - (1-u-v)\big)\big(bv - (1-u-v)\big) \\ &\quad + \big(au^2\big)^2\big(bv - (1-u-v)\big)^2 . \end{aligned}$$

Since the middle term in the right hand side equals $-\frac{1}{2}N^4F_{uv}F_{vu}$, we have

$$\frac{1}{4}N^4 d > \Big(bv^2\big(au - (1-u-v)\big) + au^2\big(bv - (1-u-v)\big)\Big)^2 \geqslant 0 .$$

Hence

$$d(\mathbf{u}) > 0 \quad \text{at any point } \mathbf{u} \neq \mathbf{U} .$$

The theoretical concepts developed in Chapter 10 are based on the assumptions that the system of conservation laws is strictly hyperbolic and genuinely nonlinear. As a consequence, the wave curves (Hugoniot locus for shocks and rarefaction curves) form a coordinate system that enables us to construct a solution of the Riemann problem satisfying the Lax entropy conditions. In the physical space the solution starts at the given left state, followed by a first wave curve to a middle state, then followed by a second wave curve to the given right state.

The occurence of the interior umbilic point $\mathbf{U}$ changes the topology of the wave curves. As discussed by ISAACSON ET AL. [40, 38], the Hugoniot locus can have disconnected branches and the rarefaction curves no longer form a coordinate system in a neighbourhood of the umbilic point. This is shown in Figure 12.9 for the symmetric case $a = b = 1$, giving $\mathbf{U} = (\frac{1}{3}, \frac{1}{3})$, see also the rarefaction curves in Figure 13.4 of Chapter 13.

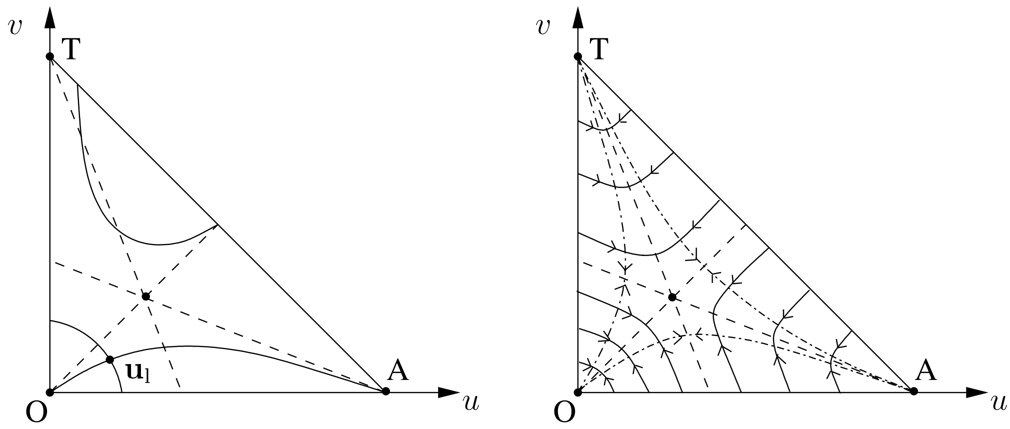


Figure 12.9. (a) *Hugoniot locus for* $\mathbf{u}_l$*;* (b) *rarefaction curves, solid lines: slow rarefactions, dash-dot lines: fast rarefactions. Arrows indicate direction of increasing characteristic speed (*λ*)*

To get some insight in the complex nature of the Hugoniot locus we consider the following example. Let $a = b = 1$ and let $\mathbf{u}_l = (\alpha, \alpha)^T$ with $0 \leqslant \alpha \leqslant \frac{1}{2}$. With $N_l = u_l^2 + (1 - u_l - v_l)^2 + v_l^2 = 2\alpha^2 + (1 - 2\alpha)^2$, we consider

$$\begin{cases} \dfrac{u^2}{N} - \dfrac{\alpha^2}{N_l} = s(u - \alpha) \; , \\[2ex] \dfrac{v^2}{N} - \dfrac{\alpha^2}{N_l} = s(v - \alpha) \; . \end{cases}$$

Eliminating the shock speed s results in the algebraic expression

$$u^2 - \frac{u - \alpha}{v - \alpha} v^2 = \frac{\alpha^2}{N_l} N \left(1 - \frac{u - \alpha}{v - \alpha} \right)$$

or

$$(v - \alpha)u^2 - (u - \alpha)v^2 = \frac{\alpha^2}{N_l} N (v - u) \; .$$

Since

$$\begin{aligned}(v-\alpha)u^2-(u-\alpha)v^2 &= \alpha(v^2-u^2)+u^2v-uv^2\\ &= \big(\alpha(v+u)-uv\big)(v-u)\ ,\end{aligned}$$

we find that the Hugoniot locus is given by

$$v = u$$

and

$$\frac{\alpha^2}{N_1}N-\alpha(u+v)+uv=0\ . \tag{12.32}$$

Using $N = u^2+(1-u-v)^2+v^2$ and setting

$$\begin{aligned} w &:= u+v\ , \quad 0 \leqslant w \leqslant 1\ ,\\ z &:= uv\ , \qquad 0 \leqslant z \leqslant \frac{1}{4}\ ,\end{aligned}$$

expression (12.32) becomes

$$z = \frac{1}{1-2\frac{\alpha^2}{N_1}}\left\{\alpha w - \frac{\alpha^2}{N_1}\big(w^2+(1-w)^2\big)\right\} =: f(w,\alpha)\ .$$

The definitions of w and z imply

$$u^2 - wu + f(w,\alpha) = 0\ .$$

By symmetry we consider only the root

$$u = \frac{w}{2}-\frac{1}{2}\sqrt{w^2-4f(w,\alpha)}\ , \tag{12.33a}$$

giving

$$v = \frac{w}{2}+\frac{1}{2}\sqrt{w^2-4f(w,\alpha)}\ . \tag{12.33b}$$

Direct computation shows:

(*i*) $f(0,\alpha) = \dfrac{-\alpha^2}{(1-2\alpha)^2} < 0$;

(*ii*) $\dfrac{\partial^2 f}{\partial w^2}(w,\alpha) = 4f(0,\alpha) < 0 \quad \text{for} \quad 0 \leqslant w \leqslant 1$;

(*iii*) $f(1,\alpha) = 0$ for $\alpha = 0, \frac{1}{3}$ and $\frac{1}{2}$;
$f(1,\alpha) > 0$ for $0 < \alpha < \frac{1}{3}$;
$f(1,\alpha) < 0$ for $\frac{1}{3} < \alpha < \frac{1}{2}$;

(*iv*) $f(\frac{1}{2},\alpha) = \frac{1}{2}f(1,\alpha)$;

(v) $w^2 - 4f(w,\alpha) = 0$ if and only if

$$w = \begin{cases} p_1(\alpha) = 2\alpha \,, \\ p_2(\alpha) = \dfrac{2\alpha}{12\alpha^2 - 4\alpha + 1} \,. \end{cases}$$

Hence for $0 < \alpha < \frac{1}{3}$, there exist $w_*(\alpha) \in (0, \frac{1}{2})$ (with $w_*(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ and $w_*(\alpha) \uparrow \frac{1}{2}$ as $\alpha \uparrow \frac{1}{3}$) such that

$$\begin{aligned} f(w,\alpha) &< 0 \quad \text{for} \quad 0 \leqslant w < w_*(\alpha) \,, \\ f(w,\alpha) &> 0 \quad \text{for} \quad w_*(\alpha) < w \leqslant 1 \,. \end{aligned}$$

For $\frac{1}{3} < \alpha < \frac{1}{2}$, there exist $\frac{1}{2} < w_*(\alpha) < w^*(\alpha) < 1$ (with $w_*(\alpha) \downarrow \frac{1}{2}$, $w^*(\alpha) \uparrow 1$ as $\alpha \downarrow \frac{1}{3}$ and as $\alpha \uparrow \frac{1}{2}$) such that

$$\begin{aligned} f(w,\alpha) &< 0 \quad \text{as} \quad 0 \leqslant w < w_*(\alpha) \quad \text{and for} \quad w^*(\alpha) < w \leqslant 1 \,, \\ f(w,\alpha) &> 0 \quad \text{as} \quad w_*(\alpha) < w < w^*(\alpha) \,. \end{aligned}$$

We have

$$\begin{aligned} p_1(\alpha) &< p_2(\alpha) \quad \text{for} \quad 0 < \alpha < \frac{1}{3} \,, \\ p_1(\alpha) &= p_2(\alpha) \quad \text{for} \quad \alpha = \frac{1}{3} \,, \\ p_1(\alpha) &> p_2(\alpha) \quad \text{for} \quad \frac{1}{3} < \alpha < \frac{1}{2} \,. \end{aligned}$$

Note the special role played by $\alpha = \frac{1}{3}$. This corresponds to $\mathbf{u}_l = (\frac{1}{3}, \frac{1}{3})^T = \mathbf{U}$, the interior umbilic point.

We now have the ingredients to classify the solutions of (12.32) and to construct the Hugoniot locus.

1. $\alpha = 0$.
   Since $f(w,0) = 0$ we have $z = u$, $v = 0$ giving $\{u = 0,\ \ 0 \leqslant v \leqslant 1\}$ and $\{0 \leqslant u \leqslant 1,\ \ v = 0\}$ as branches of the locus.

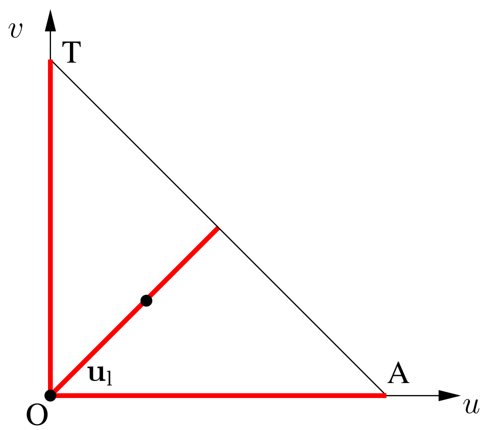


Figure 12.10. *Hugoniot locus for* $\alpha = 0$

2. $0 < \alpha < \frac{1}{3}$.
   This gives (with w as parametrization):
   No solution for $0 \leqslant w < w_*(\alpha)$, since $f < 0$ and $u < 0$;
   A unique solution for $w_*(\alpha) \leqslant w \leqslant p_1(\alpha) = 2\alpha$, with $u = 0$ as $w = w_*(\alpha)$ and $u = \alpha$ as $w = p_1(\alpha)$;
   No solution for $w_*(\alpha) < w < w^*(\alpha)$, since $w^2 - 4f < 0$;
   A unique solution for $w^*(\alpha) \leqslant w < 1$, with $u = \frac{1}{2}p_2(\alpha)$ as $w = p_2(\alpha)$ and $u = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4f(1,\alpha)}$ as $w = 1$.

   Hence this α-range yields the locus

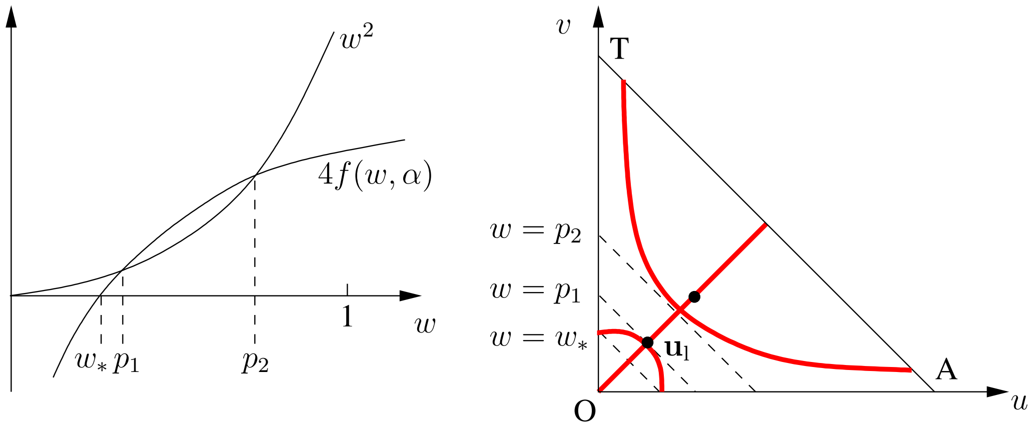


Figure 12.11. *Hugoniot locus for* $0 < \alpha < \frac{1}{3}$

Using (12.33) we find

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\mathrm{d}v}{\mathrm{d}w} \Big/ \frac{\mathrm{d}u}{\mathrm{d}w} = -1 \quad \text{as} \quad w = p_1(\alpha),\ p_2(\alpha) \quad (\alpha \neq \frac{1}{3}),$$

implying that the branches cross $u = v$ perpendicular.

3. $\alpha = \frac{1}{3}$, the umbilic point.
   Using $w_*(\frac{1}{3}) = \frac{1}{2}$, $p_1(\frac{1}{3}) = p_2(\frac{1}{3}) = \frac{2}{3}$ and $f(1, \frac{1}{3}) = 0$ (see Figure 12.12 (left)) gives:
   No solution for $0 \leqslant w < \frac{1}{2}$;
   A unique solution for $\frac{1}{2} \leqslant w \leqslant 1$, with $u = 0$ as $w = \frac{1}{2}$, $u = \frac{1}{3}$ as $w = \frac{2}{3}$ and $u = 0$ as $w = 1$.

   Since $p_1 = P_2$ we have

$$w^2 - 4f(w, \frac{1}{3}) = K\left(w - \frac{2}{3}\right)^2 \quad \text{with} \quad K = 9\,.$$

   Then (12.33) becomes

$$u = \frac{w}{2} - \frac{3}{2}\left|w - \frac{2}{3}\right|,$$
$$v = \frac{w}{2} + \frac{3}{2}\left|w - \frac{2}{3}\right|.$$

Using $\pm$ for the other branches and eliminating w, gives for the locus the explicit expression

$$u = v\ , \quad u = \frac{1-v}{2}\ , \quad v = \frac{1-u}{2}\ .$$

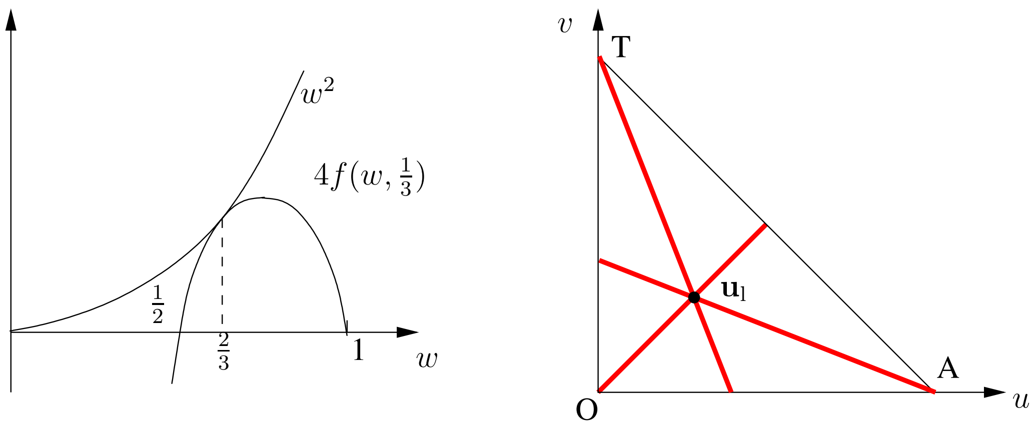


Figure 12.12. *Hugoniot locus for* $\mathbf{u}_\mathrm{l} = \mathbf{U}$*, the umbilic point*

4. $\frac{1}{3} < \alpha < \frac{1}{2}$.
   Now $p_2 < p_1$, with $p_2 < \frac{2}{3}$, and $f(\frac{1}{2}, \alpha) = \frac{1}{2} f(1, \alpha) < 0$, see Figure 12.13 (left). As in case 2 this yields the locus

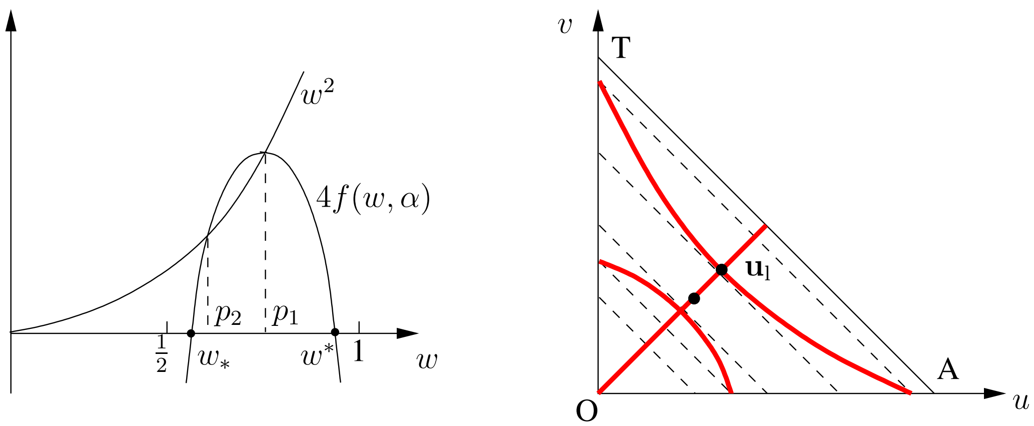


Figure 12.13. *Hugoniot locus for* $\frac{1}{3} < \alpha < \frac{1}{2}$

5. $\alpha = \frac{1}{2}$.
   Since $p_2(\alpha)$, $w_*(\alpha) \downarrow \frac{1}{2}$ and $p_1(\alpha)$, $w^*(\alpha) \uparrow 1$ as $\alpha \uparrow \frac{1}{2}$, the locus ends up as

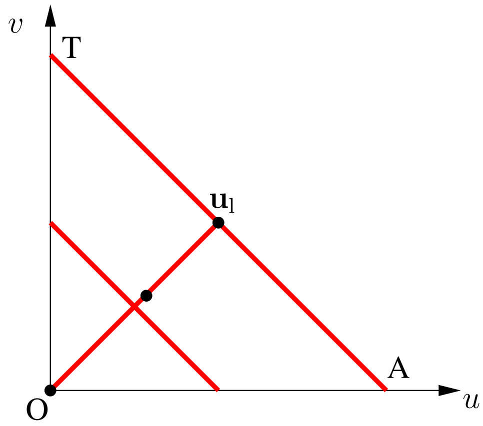


Figure 12.14. *Hugoniot locus for* $\alpha = \frac{1}{2}$

These manipulations clearly demonstrate the complexity of solving three-phase Riemann problems, even for the simplest cases. Each model has its own peculiarities and requires an approach of its own. There is little unifying theory. In the next chapter we give – as an example – the complete description of the construction of the solution of a particular three-phase Riemann problem.

13 Uniqueness conditions in a hyperbolic model for oil recovery by steamdrive

13.1 Introduction

Steamdrive, being the most important enhanced oil recovery technique, received considerable attention in the engineering literature during the past decades. As examples we mention the experimental work of KIMBER ET AL. [42], GÜMRAH ET AL. [31] and FAROUQ ALI ET AL. [22], and the modeling work of MANDL & VOLEK [49], GODDERIJ ET AL. [30] and PRATS [60]. An important characteristic of their models is the occurrence of a Steam Condensation Front (SCF) as an internal boundary between the hot steam zone and the cold liquid zone. Furthermore, in their approach, the saturation of the oil remaining behind in the steam zone does not follow from the analysis of the models, but is a-priori given as model parameter.

SHUTLER [66] proposed a relatively simple model which treats the oil saturation in the steam zone as an unknown. We explain it here in some detail because it forms the basis for our approach. In this model again a SCF is present, which separates an upstream steam zone from a downstream oil/water zone. It is assumed that all steam condenses at the SCF. The velocity of the SCF follows from a local heat balance. Because the heat capacity of the porous medium depends on the fluid saturations, there is coupling between the heat balance and the saturation equations. Although this coupling is weak, Shutler takes it into account. Because fluid saturations are constant at the SCF, he finds that its velocity is constant as well. The steam zone is considered as a zone of constant high temperature in which oil, non-condensing gas (steam) and connate water are present. In the downstream cold zone oil and water are present at the original reservoir temperature. Capillary forces are disregarded. Water and oil conservation equations applied at the SCF, combined with the Buckley–Leverett equation for gas/oil in the steam zone and for oil/water in the cold zone, lead to a complete solution of the model equations. However, the assumption that the steam zone contains connate water only is not clear. This assumption is apparently necessary to close the problem. It may also have an undesirable effect on the prediction of the efficiency of the steamdrive process. Models related to the one of Shutler have been proposed by POPE [59] and YORTSOS [79].

WINGARD & ORR [78] extended the model of Shutler to incorporate three phase flow in the steam zone. A careful inspection of their paper led us to the conclusion that the presented model cannot be used for our set of parameter values. To be precise, the upstream saturations at the SCF cannot be obtained from the upstream boundary conditions by integrating the mass balance equations. We need additional conditions at the SCF to obtain a unique solution. From a physical point of view, such conditions should originate from a detailed local model of the steam condensation process itself.

The one dimensional steamdrive model considered in this chapter unifies a hyperbolic interface model and a parabolic transition model. In the hyperbolic setting, a SCF exists and moves at a given speed through the porous medium. It separates the hot steam and the cold liquid regions. All steam condenses at the SCF and no capillary forces are present. Inspired by the work of STEWART & UDELL [68], UDELL [72] and MENEGUS & UDELL [51], we consider various local transition models. In these models, steam condenses according to a delta distribution at the SCF and fluid flow towards and from the SCF is governed by the Darcy law including capillary effects. The model equations in the transition zone are solved by the method of matched asymptotic expansions. This leads to solutions in the form of travelling waves, moving with the speed of the SCF. The conditions for such waves to exist are precisely the missing matching conditions for the saturations at the SCF. We will explicitly show how different transition models yield different saturation combinations at the SCF and consequently different solutions of the hyperbolic model. These differences are not always small. For instance when comparing the results of a transition model with constant capillary diffusion and one with Brooks–Corey three phase capillary pressures, the differences are well-noticeable and cannot be disregarded for practical purposes.

Such dependence on details of the transition model (i.e. parabolic regularization) is known to occur in systems of conservation laws. It arises in the context of transitional waves, see for instance ISAACSON ET AL. [40], [41], GUZMÁN & FAYERS [33] and GLIMM [29]. Using vanishing viscosity (in our case vanishing capillary diffusion and heat conduction) as the entropy condition for the hyperbolic system, one finds travelling waves describing the transition through the shock. If the travelling wave connects a node and a saddle (of the associated dynamical system), the resulting shock is admissible as a Lax shock, see ISAACSON ET AL. [40], [41]. If, however, the travelling wave connects two saddles, the Lax criteria fail. But the resulting shock is still admissible in the physical sense. It is called a transitional wave or shock. Saddle to saddle connections are sensitive to perturbations of the system. This explains in a unspecified way the dependence of the matching conditions at the SCF on the parameters of the transition model.

In Section 13.2 we describe the physical model. First we present the base case, with input parameters summarized in Table I and Table II. In the base case we model the transition region with constant (saturation independent) capillary diffusion, an abrupt temperature drop from steam temperature to reservoir temperature at the SCF and no steam downstream of the SCF. We also study three cases in which one of these simplifying conditions is relaxed (i) Brooks-Corey three-phase capillary pressures, (ii) an exponential temperature decline downstream the SCF, and (iii) a non-zero steam saturation downstream the SCF in the transition region. In Section 13.3 we present the analysis of the base case. In Section 13.4 we analyze the problem with different transition models. In particular we compare the results of the three cases defined in Section 13.2 with the base case. In Section 13.5 we study the variation of model parameters. There we introduce the average oil saturation in the steam zone and investigate its dependence on reservoir and fluid properties. To construct the full solution of the steamdrive problem is rather involved. Therefore we present in Section 13.5 an approximation, which is fairly straightforward to obtain. In Figure 13.13 we compare the results for the full solution and this approximation. It clearly indicates in which parameter range the approximation is acceptable for engineering purposes. We summarize our findings in Section 6 which contains the conclusions.

13.2 Physical model

Oil displacement by steamdrive through a porous medium is a complex physical process which is controlled by the steam condensation process and by viscous and capillary forces, see for instance WINGARD & ORR [78] or STEWART & UDELL [68]. In this chapter we propose a simplified approach in which all steam condenses at an a-priori known Steam Condensation Front (SCF) and in which capillary forces as well as temperature variations are disregarded except in a small neighborhood of that front. Here "small" must be understood in a suitable dimensionless context. To model this we consider a global *interface model* in which capillary forces are absent on any scale and in which the interface (SCF) separates the hot steam zone from the remainder of the reservoir. Further we consider a local *transition model* which takes capillary forces and temperature variations into account at the SCF. The transition model yields the correct matching conditions at the SCF in the hyperbolic interface model.

In modeling a one dimensional flood through a reservoir we consider the porous medium to be homogeneous, with constant porosity ϕ, and of semi-infinite extent. The multi-phase flow (oil, water, steam (gas)) through the reservoir is directed in what we choose to be the positive x−axis. Hence the phase saturations S_{o}, S_{w}, and S_{g} are functions of position x and time t only, see Figure 13.1. Initially, at $t = 0$, the reservoir contains oil and connate water: i.e. for all $x > 0$

$$S_{\mathrm{o}}(x,0) = 1 - S_{\mathrm{wc}}\ ,\ S_{\mathrm{w}}(x,0) = S_{\mathrm{wc}}\ ,\ S_{\mathrm{g}}(x,0) = 0\ . \tag{13.1}$$

From the left steam of 100 % quality is injected at rate u_{inj}: i.e at $x = 0$ and for all $t > 0$

$$S_{\mathrm{o}}(0,t) = 0\ ,\ S_{\mathrm{w}}(0,t) = S_{\mathrm{wc}}\ ,\ S_{\mathrm{g}}(0,t) = 1 - S_{\mathrm{wc}}\ . \tag{13.2}$$

In writing these initial and boundary conditions we assume that the residual oil and gas saturations are constant. Without loss of generality they are given the value zero: see also Table I, where the values of all quantities used throughout this chapter are given.

Oil and water are produced at the right, in our simplified model at $x = \infty$. All fluids, also steam, are considered incompressible. To avoid non-essential complications the thermal expansion coefficients of the fluids are taken to be zero. Heat losses to the surroundings as well as gravity effects are not considered. Furthermore, we assume that the oil is non-distillable i.e. the partial vapor pressure of the oil in the gas phase is negligible. Consequently we ignore the presence of a distillable oil bank.

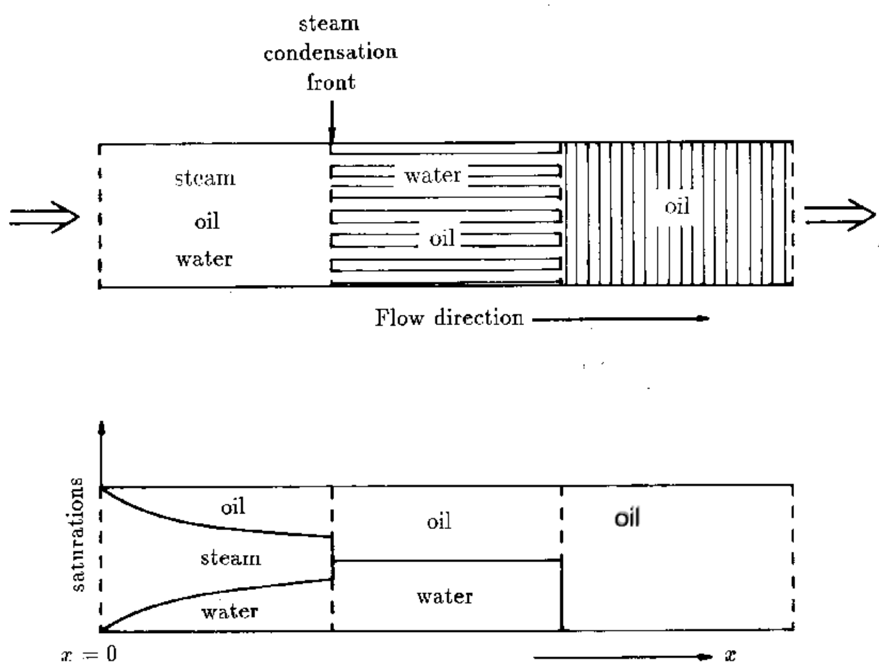


Figure 13.1. *Sketch of the one dimensional steam displacement process and the phase saturations*

13.2.1 Interface model

We distinguish two zones, see Figure 13.1, one upstream and one downstream relative to the SCF. *Upstream* is the steam zone. We assume that this zone is at constant steam temperature T_1, thus disregarding the temperature gradient as a consequence of the pressure gradient driving the fluids and the boiling point curve. Capillary forces are neglected and fluid transport is governed by Darcy's law for multi-phase flow. With the exception of Section 13.5, we use power law expressions for the relative permeabilities. In this work we keep the exponents fixed and all equal to four, see Table II. Any other choice greater than one would give the same qualitative results. *Downstream* is the liquid zone where only oil and water are present. This zone is at constant reservoir temperature T_o. Again capillary forces are disregarded and fluid transport is governed by Darcy's law for multi-phase flow. The relative permeabilities are the same as in the steam zone.

Because oil and water experience different temperatures, their viscosities $\mu_{\mathrm{i=o,w}}$ may vary substantially. To account for this we take the well-known expressions, e.g. see REID ET AL. [62] and Table I,

$$\ln \frac{\mu_\mathrm{i}}{\mu_\mathrm{r}} = a_\mathrm{i} + \frac{b_\mathrm{i}}{T} , \qquad \mathrm{i} = \mathrm{o}, \mathrm{w} . \tag{13.3}$$

The two zones are separated by the SCF. The velocity of this front v_st is determined from a local heat balance, in which the heat released by the condensing steam impinging on the SCF is equal to the amount of heat necessary to warm up the reservoir, see MANDL AND VOLEK [49]. The result is

$$v_\mathrm{st} = \frac{\rho_\mathrm{g} \Delta H \; u_\mathrm{inj}}{(\rho c)_\mathrm{r}(T_1 - T_\mathrm{o})} . \tag{13.4}$$

The symbols appearing in this expression are explained in Table I. The effective heat capacity of the reservoir includes the heat capacity of the matrix and the fluids in the pores. Variations in saturations have a relatively small effect on the effective heat capacity. This allows us to decouple the balance equations for heat and for mass. Therefore we may consider the velocity of the SCF as given.

In the interface approach the steam condenses at the SCF, $x = v_{\mathrm{st}}t$, only. Due to condensation there occurs water production Q_{w} [m^3/(m^3 s)], i.e. volume of produced water due to condensation per unit volume reservoir and per unit time, according to

$$Q_{\mathrm{w}} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}} r\delta(x - v_{\mathrm{st}}t) \; , \tag{13.5}$$

and steam loss Q_{g} [m^3/(m^3 s)], i.e. the volume of condensed steam per unit volume of reservoir per unit time, according to

$$Q_{\mathrm{g}} = r\delta(x - v_{\mathrm{st}}t) \; . \tag{13.6}$$

Here $\delta(\cdot)$ [1/m] denotes the Dirac distribution and r [m/s] the a priori unknown steam condensation rate. This factor has to be determined from the saturations at the SCF. In the absence of heat losses it does not depend on the location of the SCF. Using the values of the parameters in Table I, we find only a weak dependence of r on the saturations. Computations show that r is almost equal to the steam injection rate, see Section 13.3.3.

In order to match saturations across the SCF we need to make a detailed analysis of the possible transitions occurring there. For this we need a model which is outlined below.

13.2.2 Transition model

In the transition model we regularize the (possible) discontinuous saturations at the SCF by incorporating capillary effects. In addition we have to specify the condensation process as well as the temperature variation within the transition region. We shall first formulate a simple base case to illustrate the underlying ideas and then define three extensions.

Base case

Here we assume that the effect of capillary forces can be described in terms of a constant diffusivity D. In Section 13.3.2 we let $D \downarrow 0$ in the appropriate dimensionless setting (i.e. we let $\frac{D}{Lu_{\mathrm{inj}}} \downarrow 0$), which yields the missing matching conditions at the SCF. When $\frac{D}{Lu_{\mathrm{inj}}}$ is small, we have a small transition region which is centered at the SCF and which travels with the same velocity, see Figure 13.2. To study the saturations within the transition region we introduce the dimensionless variable

$$\xi = \frac{x - v_{\mathrm{st}}t}{L} \frac{Lu_{\mathrm{inj}}}{D} \tag{13.7}$$

and consider the blow up as $D/Lu_{\mathrm{inj}} \downarrow 0$. In terms of ξ this yields a transition region extending from $\xi = -\infty$ to $\xi = +\infty$. The corresponding limit saturations have the form of travelling waves. As $\xi \to -\infty$ the waves have to be matched with the outer saturations in the steam (hot) zone and as $\xi \to +\infty$ with the outer saturations in the liquid (cold) zone.

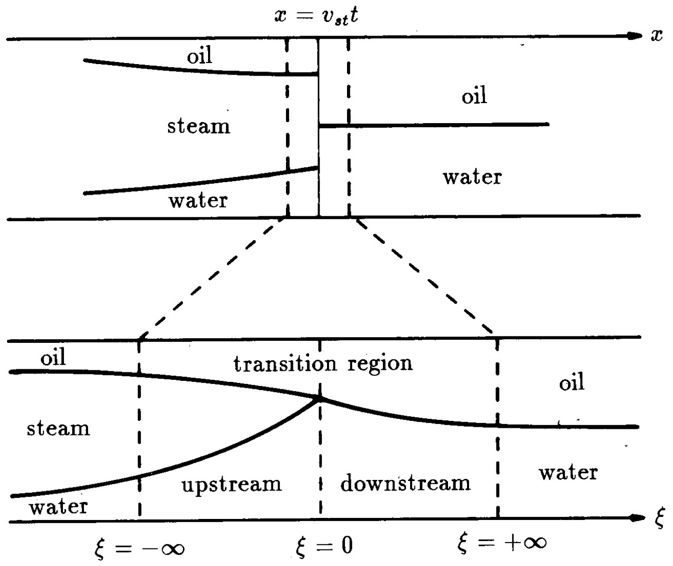


Figure 13.2. *Sketch of transition region between the steam zone and the cold zone. The transition region consists of the* SCF*, an upstream region with steam of constant temperature and a downstream region. In the base case, the downstream region has the cold reservoir temperature and no steam is present there. Possible extensions are discussed in Sections 13.2.2 - 13.2.2*

For simplicity we assume that also in the transition region the steam condenses at the SCF, where $\xi = 0$. This means that we ignore mechanisms causing a delay of the steam condensation process. Consequently two transition sub-regions can be identified: one upstream and one downstream the SCF. In both sub-regions we assume again that the temperature is constant: i.e.

$$T(\xi) = \begin{cases} T_1 & \text{for} \quad \xi < 0 \,, \\ T_\mathrm{o} & \text{for} \quad \xi > 0 \,. \end{cases} \tag{13.8}$$

We use this expression in the viscosity formula (13.3) to account for the temperature change in the transition region. Expression 13.8 describes the case where the temperature changes at a much smaller scale than the saturations. The case where temperature and saturations change at similar scales is considered in Section 13.2.2. The assumption of local thermodynamic equilibrium (T_o is much smaller than the boiling temperature) means that no steam can be present in the downstream part of the transition zone. In particular it implies

$$S_\mathrm{g}(\xi = 0) = 0 \,. \tag{13.9}$$

It turns out that this condition is needed as well to obtain a unique set of matching conditions at the SCF in the interface model. However, we will investigate cases where $S_\mathrm{g}(0) > 0$ as well.

Brooks–Corey capillary pressure diffusion

In this extension of the base case we keep (13.8) and (13.9) but we take the capillary forces more realistically into account. Clearly, this involves the introduction of three phase capillary pressures.

Since experimental data are hardly available, we assume that the oil-water capillary pressure $p_{\mathrm{o}} - p_{\mathrm{w}}$ only depends on the water saturation and the steam-oil capillary pressure $p_{\mathrm{g}} - p_{\mathrm{o}}$ only on the steam saturation (see e.g. AZIZ & SETTARI [8]). Combining these pressures an expression results for the steam-water capillary pressure $p_{\mathrm{g}} - p_{\mathrm{w}}$. Thus in this approach, three phase capillary pressures can be expressed in terms of well-known two-phase capillary pressures. The saturation dependence of the capillary pressures enters through the Leverett-functions. We write

$$P_{\mathrm{c}}^{\mathrm{ow}} = \sigma\sqrt{\frac{\phi}{k}}J^{\mathrm{ow}}(S_{\mathrm{w}}) \quad \text{and} \quad P_{\mathrm{c}}^{\mathrm{go}} = \sigma\sqrt{\frac{\phi}{k}}J^{\mathrm{go}}(S_{\mathrm{g}}) \; , \tag{13.10}$$

where we have used the fact that the interfacial tension (σ) between oil and water and between gas and oil is approximately the same. For the Leverett functions we use the empirical Brooks–Corey expressions, see for instance DULLIEN [20]. This means that J^{ow} is proportional to

$$\left(\frac{S_{\mathrm{w}} - S_{\mathrm{wc}}}{1 - S_{\mathrm{wc}}}\right)^{-1/\lambda_{\mathrm{s}}} \; , \tag{13.11}$$

where λ_{s} is a factor related to the sorting. The expression for J^{go} is obtained by substituting $S_{\mathrm{w}} = 1 - S_{\mathrm{g}}$ into (13.11).

When λ_{s} is large the capillary pressure curve is flat, meaning that the grains have approximately the same size and are well sorted. When λ_{s} is small the capillary pressure curve is steep, and the grains are badly sorted. Finally we assume that the Leverett function satisfies $J(\frac{1}{2}) = \frac{1}{2}$. For most experimental data, as in [20], indeed $0.3 < J(\frac{1}{2}) < 0.7$. All of this leads to the following expressions for the capillary pressure:

$$P_{\mathrm{c}}^{\mathrm{ow}}(S_{\mathrm{w}}) = \frac{\sigma}{2}\sqrt{\frac{\phi}{k}}\left(\frac{\frac{1}{2} - S_{\mathrm{wc}}}{1 - S_{\mathrm{wc}}}\right)^{1/\lambda_{\mathrm{s}}}\left(\frac{S_{\mathrm{w}} - S_{\mathrm{wc}}}{1 - S_{\mathrm{wc}}}\right)^{-1/\lambda_{\mathrm{s}}} \quad \text{and} \quad P_{\mathrm{c}}^{\mathrm{go}}(S_{\mathrm{g}}) = P_{\mathrm{c}}^{\mathrm{ow}}(1 - S_{\mathrm{g}}) \; . \tag{13.12}$$

In Section 4 we introduce the capillary pressure functions in the different equations. This leads to terms resembling non-linear diffusion. As a characteristic capillary diffusion number we find

$$D = \frac{\sigma\sqrt{\phi k}}{\mu_{\mathrm{o}}} \; . \tag{13.13}$$

As in the base case we investigate the process $\dfrac{D}{Lu_{\mathrm{inj}}} \downarrow 0$ to obtain matching conditions for the interface model.

Temperature variation

Here we consider constant capillary diffusion and (13.9), but we modify (13.8). To model the temperature distribution properly, one must consider the heat-balance equation in terms of the local coordinate ξ and construct a solution satisfying $T \to T_1$ as $\xi \to -\infty$ and $T \to T_{\mathrm{o}}$ as $\xi \to +\infty$. This procedure may be complicated because the coefficients in the temperature equation depend on the fluid saturations. Ignoring this dependence, MILLER [53] finds a solution of the form

$$T(\xi) = \begin{cases} T_1 & \text{for} \quad \xi < 0 \; , \\ T_{\mathrm{o}} + (T_1 - T_{\mathrm{o}})e^{-\alpha\xi} & \text{for} \quad \xi > 0 \; . \end{cases} \tag{13.14}$$

Here the constant α is the ratio of the the front velocity and the thermal conductivity in the appropriate dimensionless setting.

Positive steam saturation at SCF

Now we consider a constant capillary diffusivity and (13.8), but we modify (13.9). If we drop the assumption concerning local thermodynamic equilibrium, there is no physical reason why (13.9) would hold. In that case, steam condenses at a rate which is limited by diffusional processes in the vapor zone. Corresponding to this we construct solutions for which steam is also present in the downstream region. To obtain such solutions we have to prescribe a positive value for the steam saturation at the SCF:

$$S_{\rm g}(\xi = 0) > 0 \quad \text{(prescribed)} \ . \tag{13.15}$$

Remark 13.1. *In Section 13.5 we discuss the results of computations for the full Brooks–Corey case. There we keep (13.8) and (13.9) in the transition model, but we modify both the capillary pressure and relative permeabilities according to Brooks–Corey and Corey–Stone expressions. This is a modification of Section 13.2.2 in the sense that power law relative permeabilities are replaced by the Corey–Stone relative permeabilities, where $k_{\rm rw} = k_{\rm rw}(S_{\rm w}), k_{\rm rg} = k_{\rm rg}(S_{\rm g})$ and $k_{\rm ro} = k_{\rm ro}(S_{\rm w}, S_{\rm g})$.*

Table I, Summary of physical input parameters*

Physical quantity	*symbol*	*value*	*unit*
characteristic length	L	100	[m]
steam temperature	T_1	486	[K]
reservoir temperature	$T_{\rm o}$	313	[K]
injection rate steam	$u_{\rm inj}$	9.52 10^{-4}	[m^3/m^2/s]
steam viscosity	$\mu_{\rm g}$	1.63 10^{-5}	[Pa s]
oil viscosity at T_1	$\mu_{\rm o}(T_1)$	2.45 10^{-3}	[Pa s]
oil viscosity at $T_{\rm o}$	$\mu_{\rm o}(T_{\rm o})$	0.180	[Pa s]
water viscosity at T_1	$\mu_{\rm w}(T_1)$	1.30 10^{-4}	[Pa s]
water viscosity at $T_{\rm o}$	$\mu_{\rm w}(T_{\rm o})$	7.21 10^{-4}	[Pa s]
viscosity $\ln \mu_{\rm i}/\mu_{\rm r} = a_{\rm i} + b_{\rm i}/T$	$\mu_{\rm i}$	$\mu_{\rm i}(T)$	[Pa s]
reference viscosity	$\mu_{\rm r}$	1	[Pa s]
coefficient in oil viscosity	$a_{\rm o}$	-13.79	[$-$]
coefficient in oil viscosity	$b_{\rm o}$	3781	[K]
coefficient in water viscosity	$a_{\rm w}$	-12.06	[$-$]
coefficient in oil viscosity	$b_{\rm w}$	1509	[K]
Brooks–Corey sorting factor	$\lambda_{\rm s}$	2	[-]
enthalpy $H_2O(l)(T_{\rm o}) \to H_2O(g)(T_1)$	ΔH	2636	[kJ/kg]
effective heat capacity of rock	$(\rho c)_{\rm r}$	2029	[kJ/m^3/K]
thermal coefficient in (13.14)	α	0.017	[-]
capillary diffusion constant	D	2.2 10^{-7}	[m^2/s]
velocity SCF	$v_{\rm st}$	7.12 10^{-5}	[m/s]
porosity	ϕ	0.38	[m^3/m^3]
permeability	k	4.3 10^{-13}	[m^2]
interfacial tension	σ	30 10^{-3}	[N/m]
water density	$\rho_{\rm w}$	1000	[kg/m^3]
steam density	$\rho_{\rm g}$	10.2	[kg/m^3]
connate water saturation	$S_{\rm wc}$	0.15	[m^3/m^3]
residual gas saturation	$S_{\rm gr}$	0.0	[m^3/m^3]
residual oil saturation	$S_{\rm or}$	0.0	[m^3/m^3]

Table II, Expressions for relative permeabilities		
symbol	quantity	expression
k_{rw}	water permeability	$((S_{\mathrm{w}} - S_{\mathrm{wc}})/(1 - S_{\mathrm{wc}}))^4$
k_{ro}	oil permeability	$(S_{\mathrm{o}}/(1 - S_{\mathrm{wc}}))^4$
k_{rg}	steam permeability	$(S_{\mathrm{g}}/(1 - S_{\mathrm{wc}}))^4$

13.3 Mathematical formulation of base case

13.3.1 Interface model

The interface model described in Section 13.2.1 results in the following mass balance equations, see for instance FALLS & SCHULTE [21],

$$\phi\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial u f_{\mathrm{w}}}{\partial x} = Q_{\mathrm{w}} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}} r\delta(x - v_{\mathrm{st}}t) , \tag{13.16a}$$

$$\phi\frac{\partial S_{\mathrm{g}}}{\partial t} + \frac{\partial u f_{\mathrm{g}}}{\partial x} = -Q_{\mathrm{g}} = -r\delta(x - v_{\mathrm{st}}t) , \tag{13.16b}$$

$$\phi\frac{\partial S_{\mathrm{o}}}{\partial t} + \frac{\partial u f_{\mathrm{o}}}{\partial x} = 0 . \tag{13.16c}$$

The non-zero terms in the right side of equations (13.16a) and (13.16b) are a consequence of the steam condensation at the SCF, see also expressions (13.5) and (13.6). Except for these terms, system (13.16a)–(13.16c) consists of the standard multi-phase flow equations in which u denotes the total specific discharge and f_{i} (i=o,w,g) the fractional flow functions

$$f_{\mathrm{i}} = \frac{M_{\mathrm{oi}}k_{\mathrm{ri}}}{M_{\mathrm{ow}}k_{\mathrm{rw}} + k_{\mathrm{ro}} + M_{\mathrm{og}}k_{\mathrm{rg}}} , \tag{13.17}$$

where M_{oi} are the mobility ratio's

$$M_{\mathrm{oi}} = \frac{\mu_{\mathrm{o}}}{\mu_{\mathrm{i}}} . \tag{13.18}$$

Note that these quantities have different values up and downstream the SCF. This is due to the temperature dependence of the viscosity which enters through equations (13.3). In the interface model we will not write this dependence explicitly. Furthermore note that the specific discharge u and the steam condensation rate r are both unknown and have to be determined from the problem. However, by adding equations (13.16a)–(13.16c) and using $\sum S_{\mathrm{i}} = \sum f_{\mathrm{i}} = 1$, we find the volume balance

$$\frac{\partial u}{\partial x} = -r\left(1 - \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\right)\delta(x - v_{\mathrm{st}}t) .$$

Applying the boundary condition $u(0,t) = u_{\mathrm{inj}}$ (steam injection rate), we find upon integration

$$u = u(x,t) = u_{\mathrm{inj}} - r\left(1 - \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\right)H(x - v_{\mathrm{st}}t) , \tag{13.19}$$

*The values of the steam parameters in Table I assume a steam pressure of 20 bar. Furthermore the value of the thermal coefficient α is based on a thermal diffusivity of 9.85 10^{-7} [m^2 /s]. Note that this coefficient is proportional to the ratio of the capillary and thermal diffusivity.

where H denotes the Heaviside function: $H(s) = 0$ for $s < 0$ and $H(s) = 1$ for $s > 0$. Thus the phase saturations and the constant r have to be determined from equations (13.16a)–(13.16c), (13.19) and the initial-boundary conditions (13.1), (13.2).

Next we rewrite the equations in dimensionless form by redefining

$$S_{\mathrm{w}} := \frac{S_{\mathrm{w}} - S_{\mathrm{wc}}}{1 - S_{\mathrm{wc}}}\ , \quad S_{\mathrm{o}} := \frac{S_{\mathrm{o}}}{1 - S_{\mathrm{wc}}}\ , \quad S_{\mathrm{g}} := \frac{S_{\mathrm{g}}}{1 - S_{\mathrm{wc}}}$$
$$t := \frac{u_{\mathrm{inj}}\, t}{\phi L}\ , \qquad u := \frac{u}{u_{\mathrm{inj}}}\ , \qquad x := \frac{x}{L}\ ,$$

and by introducing the dimensionless steam condensation rate

$$\Lambda = \frac{r}{u_{\mathrm{inj}}}\ , \tag{13.20}$$

and the dimensionless SCF velocity

$$v = \frac{v_{\mathrm{st}}}{u_{\mathrm{inj}}}\phi(1 - S_{\mathrm{wc}})\ . \tag{13.21}$$

Eliminating the oil saturation by setting $S_{\mathrm{o}} = 1 - S_{\mathrm{w}} - S_{\mathrm{g}}$, we obtain the steamdrive problem:

(SD)

Find the phase saturations S_{w}, S_{g} and the condensation constant Λ such that

$$\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial u f_{\mathrm{w}}}{\partial x} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\Lambda\delta(x - vt)\ , \tag{13.22}$$

$$\frac{\partial S_{\mathrm{g}}}{\partial t} + \frac{\partial u f_{\mathrm{g}}}{\partial x} = -\Lambda\delta(x - vt)\ , \tag{13.23}$$

and

$$u = 1 - \Lambda\left(1 - \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\right) H(x - vt)\ , \tag{13.24}$$

for $x > 0$ and $t > 0$, subject to initial-boundary conditions

$$S_{\mathrm{w}}(x,0) = 0\ , \quad S_{\mathrm{g}}(x,0) = 0 \quad \text{for all } x > 0\ , \tag{13.25}$$

and

$$S_{\mathrm{w}}(0,t) = 0\ , \quad S_{\mathrm{g}}(0,t) = 1 \quad \text{for all } t > 0\ . \tag{13.26}$$

We shall consider solutions of this problem for which no steam is present in the downstream region: i.e. we pose the additional condition (as part of (SD))

$$S_{\mathrm{g}}(x,t) = 0 \quad \text{for } x > vt,\ \ t > 0\ . \tag{13.27}$$

This seems a natural condition since the temperature in this region is the cold reservoir temperature T_{o} at which no steam can survive at the current reservoir pressure. In Figure 13.3 we show the regions in which the various phases are present.

In analyzing (SD), we shall frequently represent (part of) the solution in the $(S_{\mathrm{w}}, S_{\mathrm{g}})$ plane (phase plane). Since $0 \leqslant S_{\mathrm{w}} + S_{\mathrm{g}} = 1 - S_{\mathrm{o}} \leqslant 1$, the solution is confined to the closed triangular domain $\mathcal{D}$

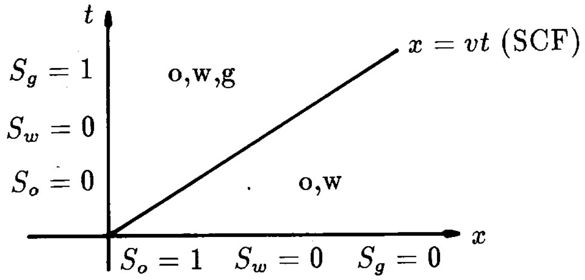


Figure 13.3. *Distribution of phases in the $x - t$ plane*

in Figure 13.4. The vertices are denoted by O = (0,0), T = (0,1) and A = (1,0). Note that any solution must pass through the points T (boundary conditions) and O (initial conditions), and must coincide with part of the S_{w}-axis (solution in the cold zone where S_{g} = 0).

In the steam zone $x < vt$, where the three phases are present and where $u = 1$ (see (13.24)), we have to solve (13.22) and (13.23), which we write in vector notation as

$$\frac{\partial \mathbf{S}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{S}) = \mathbf{0} \,. \tag{13.28}$$

Here $\mathbf{S}$ and $\mathbf{f}$ denote the column vectors $\mathbf{S} = (S_{\mathrm{w}}, S_{\mathrm{g}})^T$ and $\mathbf{f} = (f_{\mathrm{w}}, f_{\mathrm{g}})^T$. The eigenvalues λ_1 and λ_2 of the Jacobian matrix

$$D\mathbf{f} = \begin{pmatrix} f_{\mathrm{ww}} & f_{\mathrm{wg}} \\ f_{\mathrm{gw}} & f_{\mathrm{gg}} \end{pmatrix} \,, \tag{13.29}$$

where $f_{\mathrm{ij}} = \dfrac{\partial f_{\mathrm{i}}}{\partial S_{\mathrm{j}}}$ (i,j=w,g), are given by

$$\lambda_{\mathrm{k}}(\mathbf{S}) = \frac{1}{2}(f_{\mathrm{ww}} + f_{\mathrm{gg}}) + (-1)^k \frac{1}{2}\sqrt{\{(f_{\mathrm{ww}} - f_{\mathrm{gg}})^2 + 4 f_{\mathrm{wg}} f_{\mathrm{gw}}\}} \,. \tag{13.30}$$

Because we used Corey type relative permeabilities, as in Table II, the eigenvalues are real for all saturations in $\mathcal{D}$. Moreover, the system is strictly hyperbolic, with $0 \leqslant \lambda_1 < \lambda_2$, except at four points: The three vertices and one interior point. These are the umbilic points, see MARCHESIN ET AL. [50] and GUZMÁN & FAYERS [32]. There the eigenvalues are equal and the hyperbolic system degenerates. Due to the high mobility contrast, the interior umbilic point is close to O and plays no role in the analysis.

The right eigenvectors of $D\mathbf{f}$ are denoted by $\mathbf{t}_{\mathrm{k}} = \mathbf{t}_{\mathrm{k}}(\mathbf{S})$, k=1,2. A solution of (13.28), satisfying constant boundary conditions, consists in general of a combination of shock waves, constant states and rarefaction waves, see for example LAX [45], LEVEQUE [47], SMOLLER [67], or HELLFERICH [34]. Rarefaction waves are self-similar solutions depending on $\eta = x/t$ only. Considering $\mathbf{S} = \mathbf{S}(\eta)$, we find from (13.28) that they satisfy

$$-\eta\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\eta} + \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{f}(\mathbf{S}) = \mathbf{0} \tag{13.31}$$

or

$$-\eta\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\eta}+D\mathbf{f}\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\eta}=0\ . \tag{13.32}$$

in which we recognize an eigenvalue problem for the matrix $D\mathbf{f}$. Hence

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\eta}=\alpha(\eta)\mathbf{t}_{\mathrm{k}}(\mathbf{S}) \tag{13.33}$$

and

$$\eta=\lambda_{\mathrm{k}}(\mathbf{S})\ , \tag{13.34}$$

where α is an η-dependent proportionality factor which is a-priori unknown. As a consequence of (13.34) we observe that a rarefaction wave is only possible if the eigenvalue varies monotonically along its representation in the phase plane. Differentiating (13.34) with respect to η and using (13.33) gives

$$1=\nabla\lambda_{\mathrm{k}}(\mathbf{S})\cdot\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\eta}=\alpha(\eta)\nabla\lambda_{\mathrm{k}}(\mathbf{S})\cdot\mathbf{t}_{\mathrm{k}}(\mathbf{S})\ , \tag{13.35}$$

where ∇ denotes the gradient in the phase plane. Substituting this into (13.33) yields the system

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\eta}=\frac{1}{\nabla\lambda_{\mathrm{k}}(\mathbf{S})\cdot\mathbf{t}_{\mathrm{k}}(\mathbf{S})}\mathbf{t}_{\mathrm{k}}(\mathbf{S})\ , \tag{13.36}$$

as long as $\nabla\lambda_{\mathrm{k}}(\mathbf{S})\cdot\mathbf{t}_{\mathrm{k}}(\mathbf{S})\neq\mathbf{0}$ (genuine nonlinearity, LAX [45]).

If a rarefaction is to be part of the solution of (SD) we obviously want

$$\lambda_2(\mathbf{S})\leqslant v\ , \tag{13.37}$$

since otherwise the rarefaction would exceed the SCF, yielding a multi-valued solution. The region where (13.37) holds strictly is indicated in Figure 13.4 as the set $\mathcal{D}_l$ above the curve

$$l=\{(S_{\mathrm{w}},S_{\mathrm{g}}):\lambda_2(S_{\mathrm{w}},S_{\mathrm{g}})=v\}\ . \tag{13.38}$$

In spite of (13.37) we computed solutions of the system (13.36) in the full triangular domain $\mathcal{D}$. Though not strictly necessary for the analysis presented here, this gives a complete picture of the slow and fast rarefaction waves in the $S_{\mathrm{w}}-S_{\mathrm{g}}$ phase-plane. For k=2, the fast rarefactions, we solved (13.36) for $\eta<v$ and for $\eta>v$ with initial values $(S_{\mathrm{w}}(v),S_{\mathrm{g}}(v))\in l$. Computing the orbits backwards in η we found that they all reached the top T, i.e. the boundary conditions, at $\eta=0$: see Figure 13.4 where several of these fast rarefactions are shown (solid curves). The degenerate behavior of the right side of equations (13.36) causes the collapse of the orbits in the top of the triangle. This is discussed in detail by MARCHESIN ET AL. [50].

For k=1, the slow rarefactions, we solved (13.36) forwards in η with initial values taken from the segment AT. The corresponding start value of η is

$$\eta=\lambda_1(S_{\mathrm{w}},S_{\mathrm{g}})\quad\text{with}\quad(S_{\mathrm{w}},S_{\mathrm{g}})\in\mathrm{AT}\ . \tag{13.39}$$

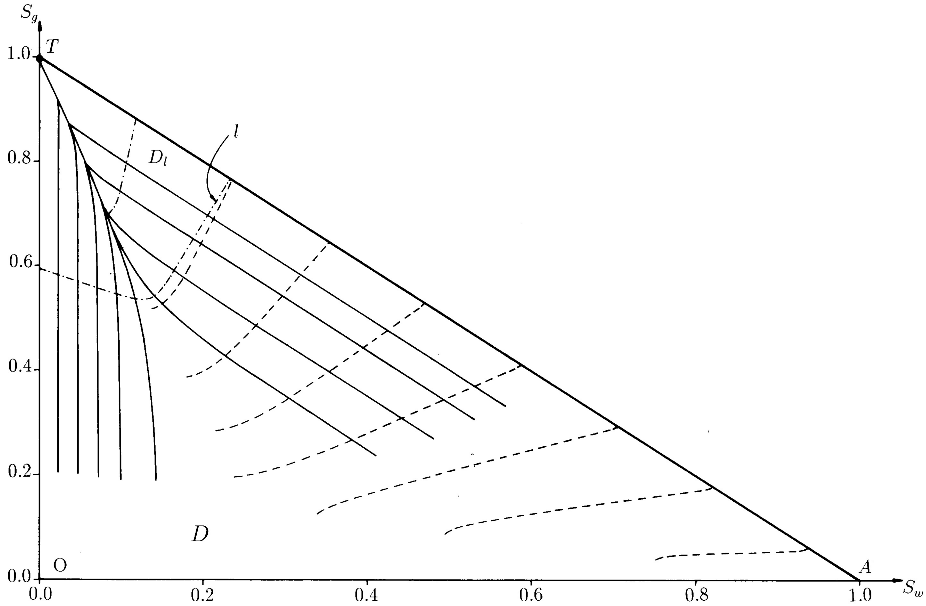


Figure 13.4. *Slow (dashed) and fast (solid) rarefactions, see (13.36). The eigenvalues of the slow rarefactions increase from the right to the left and the eigenvalues of the fast rarefactions increase from the top to the bottom*

These slow rarefactions are also shown in Figure 13.4 (dashed curves). Both slow and fast rarefactions are shown up to points where the eigenvalues reach a local extremum (along the corresponding orbits). The curves connecting these points form the inflection locus, see ISAACSON ET AL. [40]

We will not discuss the occurrence of shocks in the steam zone. Indeed, see region $\mathcal{D}_l$ in Figure 13.4, the eigenvalues for the fast rarefactions increase from top to bottom and hence shocks do not arise for our choice of boundary conditions. To find a solution of (SD) we use only fast rarefactions or constant states upstream the SCF. Later on in Section 13.3.3 where we discuss the matching conditions at the SCF, we show in fact that constant states are not allowed. Thus the solution for $x < vt$ consists of a fast rarefaction only. It connects the boundary condition $S_\mathrm{g} = 1$ to a point $(S_\mathrm{w}(v), S_\mathrm{g}(v)) \in l$ at the SCF.

If a pair $(S_\mathrm{w}^*, S_\mathrm{g}^*) \in$ AT represents a boundary condition different from (13.26), then the corresponding solution in the steam zone starts with a slow rarefaction (since $\lambda_1(S_\mathrm{w}^*, S_\mathrm{g}^*) = 0$), followed by a constant state, then followed by a fast rarefaction to match up with the SCF. This can only occur for boundary conditions above line l, provided the ensuing slow rarefaction does not intersect l before transition to the fast path.

Next we turn to the cold region downstream the SCF. Because of (13.27), only oil and water are present there. Hence we are left with the two phase Buckley–Leverett equation

$$\frac{\partial S_\mathrm{w}}{\partial t} + u^+ \frac{\partial f_\mathrm{w}}{\partial x} = 0 \quad \text{for } x > vt, \;\; t > 0 \,, \tag{13.40}$$

where u^+ denotes the downstream velocity, see (13.24),

$$u^+ = 1 - \Lambda\left(1 - \frac{\rho_{\rm g}}{\rho_{\rm w}}\right) . \tag{13.41}$$

We need to solve equation (13.40) with the a-priori unknown saturation $S_{\rm w}^+ := \lim_{x\downarrow vt} S_{\rm w}(x,t)$ along the SCF and with $S_{\rm w} = 0$ initially. Assuming $S_{\rm w}^+$ to be constant and using standard Buckley–Leverett (hyperbolic) theory, we find that the entropy solutions consist of shocks or rarefactions followed by shocks. Furthermore, only if the speed of the rarefactions or the shocks exceeds the speed of the SCF we find non-trivial solutions. With reference to Figure 13.5 this implies that

$$S_* \leqslant S_{\rm w}^+ \leqslant S^* \quad (\text{non} - \text{trivial solutions}) , \tag{13.42}$$

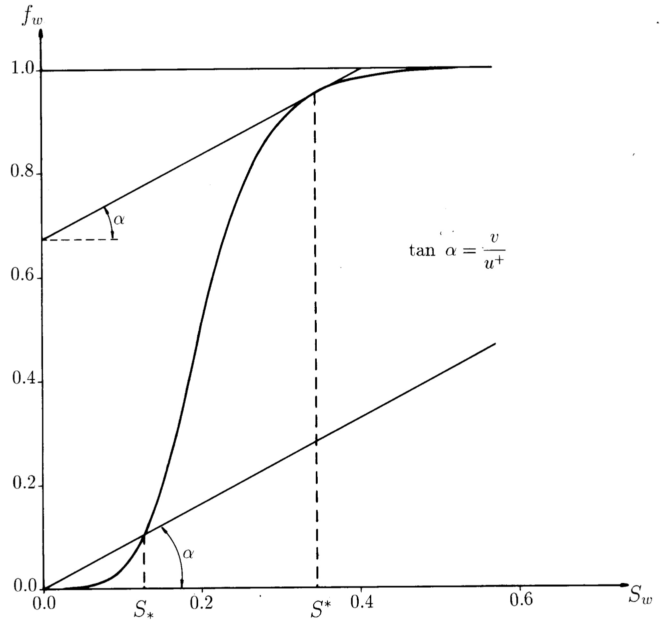


Figure 13.5. *Construction of admissible $S_{\rm w}^+$ interval*

where S_* is the (smallest) root of

$$f_{\rm w}(S_{\rm w})/S_{\rm w} = v/u^+ \tag{13.43}$$

and S^* is the largest root of

$$\frac{{\rm d}f_{\rm w}(S_{\rm w})}{{\rm d}S_{\rm w}} = v/u^+ , \tag{13.44}$$

or

$$S_{\rm w}^+ = 0 \quad (\text{trivial solution}) . \tag{13.45}$$

The value S_* corresponds to the smallest shock possible with speed $\geqslant v$, and S^* corresponds to the largest rarefaction possible with speed $\geqslant v$. Clearly (SD) does not specify the saturations at the SCF.

To find the physically correct matching condition, we need to consider the local behavior at the SCF by means of the transition model (see Section 13.2.2). We shall use the notation

$$S_{\mathrm{i}}^{+(-)} = \lim_{x \downarrow (\uparrow) vt} S_{\mathrm{i}}(x,t) \ , \quad \mathrm{i} = \mathrm{g}, \mathrm{w} \ . \tag{13.46}$$

As a consequence of (13.27) we have $S_{\mathrm{g}}^{+} = 0$.

13.3.2 Transition Model

In the transition model we include capillary forces in the form of a constant diffusivity D in all three balance equations (13.16a)–(13.16c). As we shall show below we can describe the behavior in the transition zone by the set of ordinary differential equations (13.61) and (13.62) in the upstream part and downstream part of the transition zone respectively. To recast the equations in dimensionless form we proceed as in the previous section. Introducing in addition the dimensionless diffusivity

$$\varepsilon = \frac{D}{u_{\mathrm{inj}} L} \ , \tag{13.47}$$

we obtain for the water and steam saturations

$$\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial u f_{\mathrm{w}}}{\partial x} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}} \Lambda \delta(x - vt) + \varepsilon \frac{\partial^2 S_{\mathrm{w}}}{\partial x^2} \ , \tag{13.48}$$

$$\frac{\partial S_{\mathrm{g}}}{\partial t} + \frac{\partial u f_{\mathrm{g}}}{\partial x} = - \ \Lambda \delta(x - vt) + \varepsilon \frac{\partial^2 S_{\mathrm{g}}}{\partial x^2} \ , \tag{13.49}$$

where again u and Λ satisfy (13.24). Here ε is a small number which we later send to zero. Using the values from Table I we find as a typical value $\varepsilon = 2.31 \ 10^{-6}$.

Next consider the stretched moving coordinate (see also (13.7))

$$\xi = \frac{x - vt}{\varepsilon} \ . \tag{13.50}$$

Regarding S_{w} and S_{g} as functions of ξ and t, we find instead of (13.48)–(13.49) the equations

$$\varepsilon \frac{\partial S_{\mathrm{w}}}{\partial t} - v \frac{\partial S_{\mathrm{w}}}{\partial \xi} + \frac{\partial u f_{\mathrm{w}}}{\partial \xi} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}} \Lambda \delta(\xi) + \frac{\partial^2 S_{\mathrm{w}}}{\partial \xi^2} \ , \tag{13.51}$$

$$\varepsilon \frac{\partial S_{\mathrm{g}}}{\partial t} - v \frac{\partial S_{\mathrm{g}}}{\partial \xi} + \frac{\partial u f_{\mathrm{g}}}{\partial \xi} = - \ \Lambda \delta(\xi) + \frac{\partial^2 S_{\mathrm{g}}}{\partial \xi^2} \ . \tag{13.52}$$

For ε small, in fact letting $\varepsilon \downarrow 0$, we find to leading order

$$S_{\mathrm{i}}(\xi, t) = S_{\mathrm{i}}(\xi) \ , \quad \mathrm{i} = \mathrm{w}, \mathrm{g} \ , \tag{13.53}$$

where the travelling wave type transition saturations satisfy

$$-v \frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} + \frac{\mathrm{d}u f_{\mathrm{w}}}{\mathrm{d}\xi} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}} \Lambda \delta(\xi) + \frac{\mathrm{d}^2 S_{\mathrm{w}}}{\mathrm{d}\xi^2} \ , \tag{13.54}$$

$$-v \frac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} + \frac{\mathrm{d}u f_{\mathrm{g}}}{\mathrm{d}\xi} = - \ \Lambda \delta(\xi) + \frac{\mathrm{d}^2 S_{\mathrm{g}}}{\mathrm{d}\xi^2} \ , \tag{13.55}$$

for $-\infty < \xi < \infty$. These equations imply

$$-vS_{\mathrm{w}} + uf_{\mathrm{w}} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\Lambda H(\xi) + \frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} + C_1 \, , \tag{13.56}$$

$$-vS_{\mathrm{g}} + uf_{\mathrm{g}} = - \ \Lambda H(\xi) + \frac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} + C_2 \, , \tag{13.57}$$

where C_1 and C_2 are constants of integration. Because the base case temperature satisfies (13.8), we find that the water and oil viscosity and hence the mobility ratios M_{ow} and M_{og} have different values for $\xi > 0$ and $\xi < 0$. This means that the fractional flow functions in equations (13.56) and (13.57) also have a discontinuous ξ-dependence: $f_{\mathrm{i}} = f_{\mathrm{i}}^{\mathrm{r}}(S_{\mathrm{w}}, S_{\mathrm{g}})$ for $\xi > 0$ and $f_{\mathrm{i}} = f_{\mathrm{i}}^{\mathrm{l}}(S_{\mathrm{w}}, S_{\mathrm{g}})$ for $\xi < 0$.

We solve the transition saturation equations subject to the boundary conditions (13.46):

$$S_{\mathrm{w}}(-\infty) = S_{\mathrm{w}}^{-} \quad , \quad S_{\mathrm{g}}(-\infty) = S_{\mathrm{g}}^{-} \tag{13.58}$$

and

$$S_{\mathrm{w}}(+\infty) = S_{\mathrm{w}}^{+} \quad , \quad S_{\mathrm{g}}(+\infty) = 0 \, . \tag{13.59}$$

Letting $\xi \to \pm\infty$ in (13.56) and (13.57) yields the Rankine–Hugoniot condition

$$(\mathrm{RH}) \begin{cases} u^{+} f_{\mathrm{w}}^{+} - vS_{\mathrm{w}}^{+} & = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\Lambda + f_{\mathrm{w}}^{-} - vS_{\mathrm{w}}^{-} \\ 0 & = -\Lambda + f_{\mathrm{g}}^{-} - vS_{\mathrm{g}}^{-} \, , \end{cases} \tag{13.60}$$

where $f_{\mathrm{i}}^{-} = f_{\mathrm{i}}^{\mathrm{l}}(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-})$, $f_{\mathrm{w}}^{+} = f_{\mathrm{w}}^{\mathrm{r}}(S_{\mathrm{w}}^{+}, S_{\mathrm{g}}^{+})$ and v the shock speed.

We will formulate conditions, in addition to (13.37), (13.42) and (13.60), which enable us to select a unique set of boundary values (13.58), (13.59). These conditions are related to the solvability of the boundary value problem (13.56)–(13.59). To investigate this we consider two sub-problems. Eliminating the constants C_1 and C_2 from equations (13.56) and (13.57), we consider for $\xi < 0$ the boundary value problem

$$(\mathrm{P}^{\mathrm{l}}) \begin{cases} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} & = & f_{\mathrm{w}}^{\mathrm{l}} - vS_{\mathrm{w}} - (f_{\mathrm{w}}^{-} - vS_{\mathrm{w}}^{-}) \, , \\ \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} & = & f_{\mathrm{g}}^{\mathrm{l}} - vS_{\mathrm{g}} - (f_{\mathrm{g}}^{-} - vS_{\mathrm{g}}^{-}) \, , \\ S_{\mathrm{w}}(-\infty) & = & S_{\mathrm{w}}^{-} \, , \ S_{\mathrm{w}}(0) = S_{\mathrm{w}}^{\mathrm{l}} \, , \\ S_{\mathrm{g}}(-\infty) & = & S_{\mathrm{g}}^{-} \, , \ S_{\mathrm{g}}(0) = S_{\mathrm{g}}^{\mathrm{l}} \, , \end{cases} \tag{13.61}$$

and for $\xi > 0$

$$(\mathrm{P}^{\mathrm{r}}) \begin{cases} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} & = & u^{+} f_{\mathrm{w}}^{\mathrm{r}} - vS_{\mathrm{w}} - (u^{+} f_{\mathrm{w}}^{+} - vS_{\mathrm{w}}^{+}) \, , \\ \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} & = & u^{+} f_{\mathrm{g}}^{\mathrm{r}} - vS_{\mathrm{g}} \, , \\ S_{\mathrm{w}}(+\infty) & = & S_{\mathrm{w}}^{+} \, , \ S_{\mathrm{w}}(0) = S_{\mathrm{w}}^{\mathrm{r}} \, , \\ S_{\mathrm{g}}(+\infty) & = & 0 \, , \ S_{\mathrm{g}}(0) = S_{\mathrm{g}}^{\mathrm{r}} \, , \end{cases} \tag{13.62}$$

where we have used that $u^+ f_{\rm g}^+ - vS_{\rm g}^+ = 0$. We need to find such boundary values $S_{\rm w}^-$, $S_{\rm g}^-$ and $S_{\rm w}^+$, so that the subproblems $(\mathrm{P}^{\rm l})$ and $(\mathrm{P}^{\rm r})$ admit a solution with $S_{\rm w}^{\rm l} = S_{\rm w}^{\rm r}$ and $S_{\rm g}^{\rm l} = S_{\rm g}^{\rm r}$. For that choice we have continuous transition saturations that satisfy equations (13.56) and (13.57). Only if we make the additional assumption (13.9) about the value of the steam saturation at the SCF, we find unique values $S_{\rm w}^-$, $S_{\rm g}^-$ and $S_{\rm w}^+$. This will be explained in the next section.

13.3.3 Matching Conditions

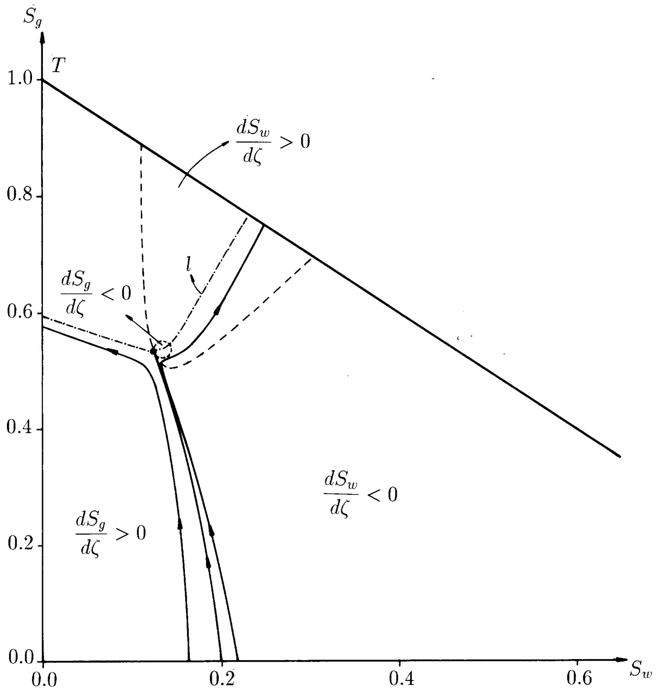
We first consider $(\mathrm{P}^{\rm l})$. To determine the nature of the equilibrium point $(S_{\rm w}^-, S_{\rm g}^-)$ we compute the eigenvalues $e_{\rm k}$ (k=1,2) of the linearized system at that point. This yields

$$e_{\rm k} = \lambda_{\rm k} - v \; , \tag{13.63}$$

where $\lambda_{\rm k}$ are the eigenvalues of the Jacobian matrix $D\mathbf{f}$, see (13.30). Consequently, if we take $(S_{\rm w}^-, S_{\rm g}^-) \in \mathcal{D}_l$, we find that $e_1 < e_2 < 0$. This means that no non-trivial orbit is possible that ends up in $(S_{\rm w}^-, S_{\rm g}^-)$ as $\xi \to -\infty$. Combining this information with (13.37) we find as remaining possibility $(S_{\rm w}^-, S_{\rm g}^-) \in l$: in other words, the saturations at the upstream side of the SCF must satisfy the condition

$$\lambda_2(S_{\rm w}^-, S_{\rm g}^-) = v \; , \tag{13.64}$$

implying that $(S_{\rm w}^-, S_{\rm g}^-)$ is a non-hyperbolic saddle for problem 13.61 with $e_1 < e_2 = 0$. Given a pair $(S_{\rm w}^-, S_{\rm g}^-)$ satisfying this condition, we find the orbit that represents the solution of $(\mathrm{P}^{\rm l})$ by the following shooting procedure. Let $S_{\rm g}(0)$ be the prescribed value of the steam saturation at the SCF. We fix $S_{\rm g}^{\rm l} = S_{\rm g}(0)$ in $(\mathrm{P}^{\rm l})$ and take $S_{\rm w}^{\rm l}$ as a shooting parameter: that is we solve the equations in $(\mathrm{P}^{\rm l})$ by a fourth order Runge–Kutta procedure in negative ξ-direction with start values $(S_{\rm w}^{\rm l}, S_{\rm g}^{\rm l})$. The corresponding orbit will deflect either to the left or to the right, see Figure 13.6 (top). Applying the bisection method, one finds after a number of iterations an accurate approximation of the water saturation at the origin $S_{\rm w}(0) = S_{\rm w}^{\rm l}$ for which a solution exists at the given values of $S_{\rm w}^-, S_{\rm g}^-$ and $S_{\rm g}(0)$.



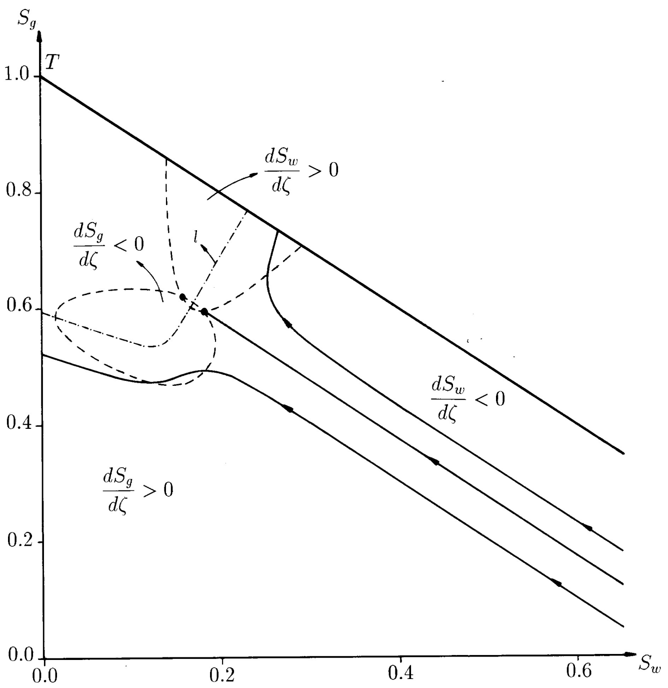


Figure 13.6. *Shooting procedure to solve* (P^1). *Here* $S_g(0) = 0$ *Top: flow diagram for* $(S_w^-, S_g^-) \in l$. *Bottom: flow diagram for* $(S_w^-, S_g^-) \in \mathcal{D}_l$. *The dots indicate the location of equilibrium points. The orbits are pointing in negative* ξ *direction and* $\zeta = -\xi$

At this point it is instructive to consider the dynamics of solutions in the saturation triangle more closely. Because we solve the equations in the negative ξ-direction, we put

$$\zeta = -\xi \tag{13.65}$$

and consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\zeta} &= (f_{\mathrm{w}}^{-} - vS_{\mathrm{w}}^{-}) - f_{\mathrm{w}}^{\mathrm{l}} + vS_{\mathrm{w}} \ , \\ \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\zeta} &= (f_{\mathrm{g}}^{-} - vS_{\mathrm{g}}^{-}) - f_{\mathrm{g}}^{\mathrm{l}} + vS_{\mathrm{g}} \ , \\ S_{\mathrm{w}}(0) &= S_{\mathrm{w}}^{\mathrm{l}} \ , \ \ S_{\mathrm{g}}(0) = S_{\mathrm{g}}^{l} \ . \end{cases} \tag{13.66}$$

The qualitative behavior of orbits is determined by the location of equilibrium points and curves where either $\frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\zeta} = 0$ or $\frac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\zeta} = 0$. This is shown in Figure 13.6 for two locations of $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-})$. In the top figure we have chosen $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-}) \in l$. The location of the curves where either $\frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\zeta}$ =0 or $\frac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\zeta}$ =0 suggests the existence of only one equilibrium point being $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-})$. Three orbits are shown in this figure, all originating from the base line $S_{\mathrm{g}} = 0$: one deflects to the left and one deflects to the right of the equilibrium. The middle orbit approximates the solution that reaches $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-})$ as $\zeta \to \infty$. In the bottom figure we have chosen $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-}) \in \mathcal{D}_l$. The location of the separation curves now suggests the existence of two equilibria: one inside $\mathcal{D}_l$, being the chosen $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-})$ and one outside $\mathcal{D}_l$. Observe from the sign conditions that no orbit can reach $(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-})$ as $\zeta \to \infty$. This corresponds to the earlier observation about the negative sign of the eigenvalues of the linearized system near that point.

Let us now introduce the additional hypothesis (13.9), expressing that also in the transition zone the steam saturation vanishes at the SCF:

$$S_{\mathrm{g}}(0) = S_{\mathrm{g}}^{\mathrm{l}} = S_{\mathrm{g}}^{\mathrm{r}} = 0 \ . \tag{13.67}$$

Using this assumption we propose the following procedure for $(\mathrm{P}^{\mathrm{l}})$. Choose S_{w}^{-}, find the corresponding S_{g}^{-} so that (13.64) holds and apply the above described shooting procedure with (13.67) to find the water saturation at the SCF. This yields $S_{\mathrm{w}}^{\mathrm{l}}$ as a function of S_{w}^{-}. With values taken from Table I, we computed this function and the result is shown in Figure 13.7. Note that $S_{\mathrm{w}}^{\mathrm{l}}$ depends continuously and monotonically on S_{w}^{-} and that $S_{\mathrm{w}}^{\mathrm{l}}$ =0 whenever S_{w}^{-} =0.

Next we consider $(\mathrm{P}^{\mathrm{r}})$. To prove existence of a continuous travelling wave we want to express the water saturation just downstream the SCF in terms of S_{w}^{-} as well, (curve $S_{\mathrm{w}}^{\mathrm{r}}(1)$ in Figure 13.7). To establish this we first need to express S_{w}^{+} in terms of S_{w}^{-}. This we get by combining $S_{\mathrm{g}}^{+} = 0$, expression (13.64) and the Rankine–Hugoniot condition (13.60). Computations show, see Figure 13.8, that given any S_{w}^{-} there are two possible values for S_{w}^{+}. However, in view of (13.42), we must restrict ourselves to the lower branch in Figure 13.8, which is a monotonically decreasing function of S_{w}^{-}. Note that S^* and S_* vary slightly with S_{w}^{-}. This dependence enters through u^+.

How to obtain $S_{\mathrm{w}}^{\mathrm{r}}(1)$ in Figure 13.7? As a result of (13.67) we find $S_{\mathrm{g}}(\xi) = 0$ for all $\xi \geqslant 0$. Therefore we only need to consider the S_{w}-equation in $(\mathrm{P}^{\mathrm{r}})$, where we use $T = T_o$ in the coefficients. Writing the water equation as

$$\frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} = F_{\mathrm{w}}(S_{\mathrm{w}}) = u^+ f_{\mathrm{w}}^{\mathrm{r}}(S_{\mathrm{w}}) - vS_{\mathrm{w}} - \{u^+ f_{\mathrm{w}}^{\mathrm{r}}(S_{\mathrm{w}}^{+}) - vS_{\mathrm{w}}^{+}\} \ , \tag{13.68}$$

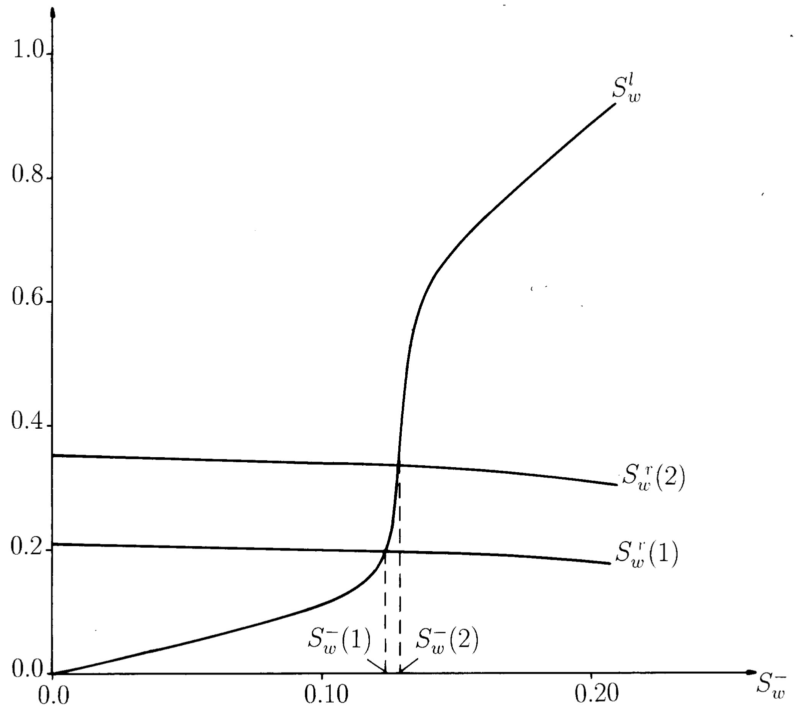


Figure 13.7. *Water saturation at the* SCF *as a function of S_w^-. Curve S_w^r(1) is computed with the base case temperature (13.8) in the transition zone. Curve $S_\mathrm{w}^\mathrm{r}(2)$ is computed with temperature (13.14) in the transition zone, see Section 13.4.2*

one easily verifies, as a consequence of (13.42) and (13.44) that $F_\mathrm{w}(S_\mathrm{w}) > 0$ for $S_\mathrm{w} > S_\mathrm{w}^+$ and $F_\mathrm{w}(S_\mathrm{w}) < 0$ for $S_\mathrm{w} < S_\mathrm{w}^+$. This implies that the only solution possible is

$$S_\mathrm{w}(\xi) = S_\mathrm{w}^+ \quad \text{for} \quad \text{all} \quad \xi \geqslant 0 \, . \tag{13.69}$$

Consequently S_w^r = S_w^+. Therefore the lower branch in Figure 13.8 also appears as $S_\mathrm{w}^\mathrm{r}(1)$ in Figure 13.7. By the monotonicity of the curves we find exactly one intersection point at S_w^- = $S_\mathrm{w}^-(1)$. At this point the values of S_w^l and S_w^r are the same, implying continuous water saturation in the transition model. The corresponding values for S_g^-, S_w^+ and Λ are found from (13.64), Figure 13.7 and (13.60). The result is:

$$S_\mathrm{w}^- = S_\mathrm{w}^-(1) = 0.1240 \, , \quad S_\mathrm{g}^- = 0.5339 \, , \quad S_\mathrm{w}^+ = 0.2014 \, , \quad \Lambda = 0.9856 \, , \tag{13.70}$$

implying that the steam condensation rate r is approximately equal to the steam injection rate u_inj. The S_w^+-value is such that downstream the SCF the solution of equation (13.40) consists of a shock only. The composite solution as a path in the saturation-temperature space is shown as curve 1 in Figure 13.9. Note that the transition saturations are monotone functions of ξ : S_g is decreasing, while S_w is increasing. In Figure 13.10 we show the saturations as a function of $\eta = x/t$. This concludes the analysis of the base case.

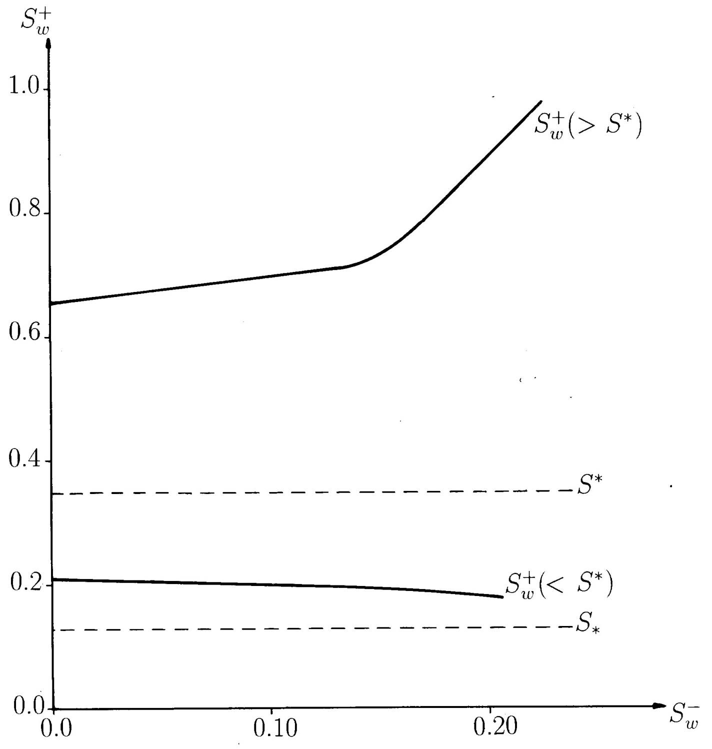


Figure 13.8. *Possible saturation combinations (S_w^-, S_w^+) satisfying the Rankine–Hugoniot conditions (13.60) and condition (13.64)*

Remark 13.2. *The solution in the transition region defines a viscous profile for the hyperbolic problem related to the base case. The constructed travelling wave connects the non-hyperbolic saddle at (S_w^-, S_g^-), located on the curve l, and the point $(S_w^+, S_g^+ = 0)$. This point is a saddle as well, because the eigenvalues satisfy $\lambda_1 = 0$ and $\lambda_2 > v$. Hence $e_1 = -v < 0$ and $e_2 > 0$. Thus the saturation shock at the* SCF *is a transitional shock in the sense of* ISAACSON ET AL. [40].

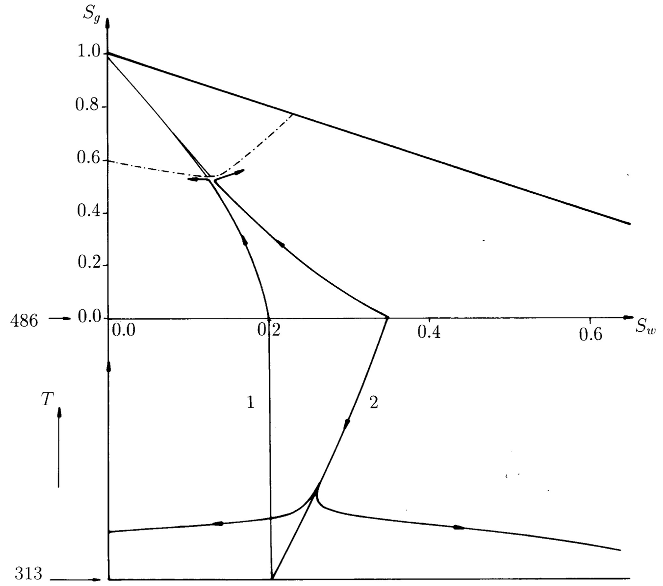


Figure 13.9. *Composite solution as path in the phase-temperature space. Curve 1 reflects the base case, in which the transition temperature is piecewise constant. Curve 2 reflects the continuously varying temperature transition as given by (13.14). Here arrows on the three orbits are pointing in the direction of the shooting procedures*

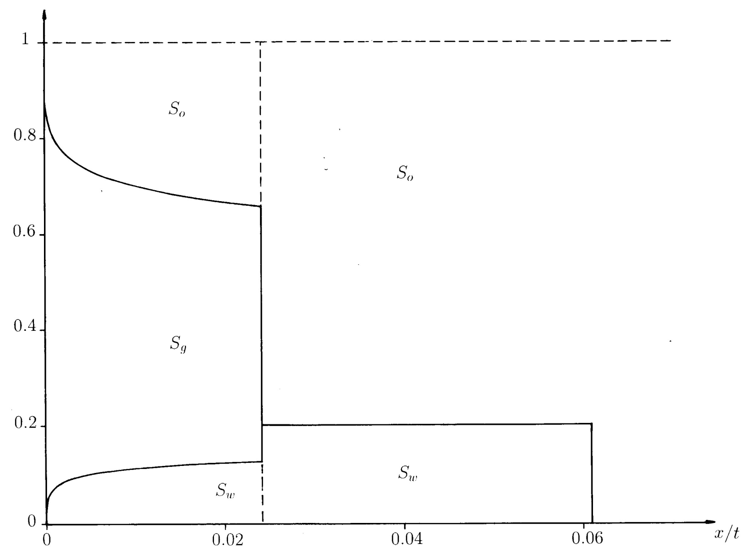


Figure 13.10. *Saturation distribution as a function of* $\eta = x/t$

13.4 Different Transition Models

In this section we investigate the relation between the transition model and the matching condition at the SCF in the global interface model.

13.4.1 Brooks-Corey capillary pressure diffusion

To incorporate the capillary pressure expressions (13.10) into the mathematical formulation of the base case, we start from Darcy's law for the individual phases

$$u_{\mathrm{i}} = -k\frac{k_{\mathrm{ri}}}{\mu_{\mathrm{i}}}\frac{\partial p_{\mathrm{i}}}{\partial x} \ , \tag{13.71}$$

and use the definitions

$$P_{\mathrm{c}}^{\mathrm{ow}} = p_{\mathrm{o}} - p_{\mathrm{w}} \ , \quad P_{\mathrm{c}}^{\mathrm{go}} = p_{\mathrm{g}} - p_{\mathrm{o}} \ , \quad P_{\mathrm{c}}^{\mathrm{gw}} = p_{\mathrm{g}} - p_{\mathrm{w}} \ , \tag{13.72}$$

to eliminate the pressures from the phase velocities. This gives

$$u_{\mathrm{w}} = uf_{\mathrm{w}} + f_{\mathrm{w}}k\frac{k_{\mathrm{ro}}}{\mu_{\mathrm{o}}}\frac{\partial P_{\mathrm{c}}^{\mathrm{ow}}}{\partial x} + f_{\mathrm{w}}k\frac{k_{\mathrm{rg}}}{\mu_{\mathrm{g}}}\frac{\partial P_{\mathrm{c}}^{\mathrm{gw}}}{\partial x} \ , \tag{13.73a}$$

$$u_{\mathrm{o}} = uf_{\mathrm{o}} - f_{\mathrm{o}}k\frac{k_{\mathrm{rw}}}{\mu_{\mathrm{w}}}\frac{\partial P_{\mathrm{c}}^{\mathrm{ow}}}{\partial x} + f_{\mathrm{o}}k\frac{k_{\mathrm{rg}}}{\mu_{\mathrm{g}}}\frac{\partial P_{\mathrm{c}}^{\mathrm{go}}}{\partial x} \ , \tag{13.73b}$$

$$u_{\mathrm{g}} = uf_{\mathrm{g}} - f_{\mathrm{g}}k\frac{k_{\mathrm{ro}}}{\mu_{\mathrm{o}}}\frac{\partial P_{\mathrm{c}}^{\mathrm{go}}}{\partial x} - f_{\mathrm{g}}k\frac{k_{\mathrm{rw}}}{\mu_{\mathrm{w}}}\frac{\partial P_{\mathrm{c}}^{\mathrm{gw}}}{\partial x} \ , \tag{13.73c}$$

where the total discharge u is given by (13.19) and the fractional flow functions f_{i} by (13.17), with power law relative permeabilities. Substituting these velocities into the phase balance equations and eliminating, as before, the oil saturation yields the modified transition equations for S_{w} and S_{g}. As in Section 13.3, we recast the equations in dimensionless form to obtain

$$\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial uf_{\mathrm{w}}}{\partial x} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\Lambda\delta(x-vt) - \varepsilon\frac{\partial}{\partial x}\left\{f_{\mathrm{w}}(k_{\mathrm{ro}} + k_{\mathrm{rg}}M_{\mathrm{og}})\frac{\partial J^{\mathrm{ow}}}{\partial x} + f_{\mathrm{w}}k_{\mathrm{rg}}M_{\mathrm{og}}\frac{\partial J^{\mathrm{go}}}{\partial x}\right\} \ , \tag{13.74}$$

$$\frac{\partial S_{\mathrm{g}}}{\partial t} + \frac{\partial uf_{\mathrm{g}}}{\partial x} = -\ \Lambda\delta(x-vt) + \varepsilon\frac{\partial}{\partial x}\left\{f_{\mathrm{g}}(k_{\mathrm{ro}} + k_{\mathrm{rw}}M_{\mathrm{ow}})\frac{\partial J^{\mathrm{go}}}{\partial x} + f_{\mathrm{g}}k_{\mathrm{rw}}M_{\mathrm{ow}}\frac{\partial J^{\mathrm{ow}}}{\partial x}\right\} \ , \tag{13.75}$$

where we have used $P_{\mathrm{c}}^{\mathrm{gw}} = P_{\mathrm{c}}^{\mathrm{go}} + P_{\mathrm{c}}^{\mathrm{ow}}$ and expressions 13.10 for $P_{\mathrm{c}}^{\mathrm{go}}$ and $P_{\mathrm{c}}^{\mathrm{ow}}$. The Leverett functions follow from (13.10) and (13.12) and the dimensionless number ε results from (13.47) and (13.13):

$$\varepsilon = \frac{D}{u_{\mathrm{inj}}L} = \frac{\sigma\sqrt{\phi k}}{\mu_o u_{\mathrm{inj}}L} \ . \tag{13.76}$$

Note that ε is related to the capillary number (capillary forces / viscous forces). Since we have assumed that $J^{\mathrm{ow}} = J^{\mathrm{ow}}(S_{\mathrm{w}})$ and $J^{\mathrm{og}} = J^{\mathrm{og}}(S_{\mathrm{g}})$, we give equations (13.74) and (13.75) the more convenient form

$$\frac{\partial S_{\mathrm{w}}}{\partial t} + \frac{\partial uf_{\mathrm{w}}}{\partial x} = \frac{\rho_{\mathrm{g}}}{\rho_{\mathrm{w}}}\Lambda\delta(x-vt) + \varepsilon\frac{\partial}{\partial x}\left\{D_{\mathrm{ww}}\frac{\partial S_{\mathrm{w}}}{\partial x} + D_{\mathrm{wg}}\frac{\partial S_{\mathrm{g}}}{\partial x}\right\} \ , \tag{13.77}$$

$$\frac{\partial S_{\mathrm{g}}}{\partial t} + \frac{\partial uf_{\mathrm{g}}}{\partial x} = -\ \Lambda\delta(x-vt) + \varepsilon\frac{\partial}{\partial x}\left\{D_{\mathrm{gw}}\frac{\partial S_{\mathrm{w}}}{\partial x} + D_{\mathrm{gg}}\frac{\partial S_{\mathrm{g}}}{\partial x}\right\} \ . \tag{13.78}$$

These equations replace the base case equations (13.48) and (13.49). We now proceed as in Section (13.3.2). That is we introduce the scaled travelling wave coordinate ξ in equations (13.77) and (13.78) and assume travelling wave type profiles for the solutions. Integrating the resulting ordinary differential equations and applying boundary conditions (13.58), (13.59) yields the same Rankine–Hugoniot conditions as before. Instead of subproblems $(\mathrm{P}^{\mathrm{l}})$ and $(\mathrm{P}^{\mathrm{r}})$, we now obtain for $\xi < 0$

$$(\mathrm{Q}^{\mathrm{l}}) \begin{cases} D^{\mathrm{l}}_{\mathrm{ww}} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} + D^{\mathrm{l}}_{\mathrm{wg}} \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} = f^{\mathrm{l}}_{\mathrm{w}} - vS_{\mathrm{w}} - (f^-_{\mathrm{w}} - vS^-_{\mathrm{w}}) \ , \\ D^{\mathrm{l}}_{\mathrm{gw}} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} + D^{\mathrm{l}}_{\mathrm{gg}} \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} = f^{\mathrm{l}}_{\mathrm{g}} - vS_{\mathrm{g}} - (f^-_{\mathrm{g}} - vS^-_{\mathrm{g}}) \ , \\ S_{\mathrm{w}}(-\infty) = S^-_{\mathrm{w}} \ , \quad S_{\mathrm{w}}(0) = S^{\mathrm{l}}_{\mathrm{w}} \ , \\ S_{\mathrm{g}}(-\infty) = S^-_{\mathrm{g}} \ , \quad S_{\mathrm{g}}(0) = 0 \ , \end{cases} \tag{13.79}$$

and for $\xi > 0$

$$(\mathrm{Q}^{\mathrm{r}}) \begin{cases} D^{\mathrm{r}}_{\mathrm{ww}} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} + D^{\mathrm{r}}_{\mathrm{wg}} \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} = u^+ f^{\mathrm{r}}_{\mathrm{w}} - vS_{\mathrm{w}} - (u^+ f^+_{\mathrm{w}} - vS^+_{\mathrm{w}}) \ , \\ D^{\mathrm{r}}_{\mathrm{gw}} \dfrac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} + D^{\mathrm{r}}_{\mathrm{gg}} \dfrac{\mathrm{d}S_{\mathrm{g}}}{\mathrm{d}\xi} = u^+ f^{\mathrm{r}}_{\mathrm{g}} - vS_{\mathrm{g}} \ , \\ S_{\mathrm{w}}(+\infty) = S^+_{\mathrm{w}} \ , \quad S_{\mathrm{w}}(0) = S^{\mathrm{r}}_{\mathrm{w}} \ , \\ S_{\mathrm{g}}(+\infty) = 0 \ , \quad S_{\mathrm{g}}(0) = 0 \ , \end{cases} \tag{13.80}$$

where we have used condition (13.67). The upper indices in the diffusion coefficients relate to the temperature difference across the SCF. The properties of the nonlinear functions imply (for $\mathrm{j} = \mathrm{l}, \mathrm{r}$)

$$D^{\mathrm{j}}_{\mathrm{ww}}, D^{\mathrm{j}}_{\mathrm{gg}} > 0 \quad \text{and} \quad D^{\mathrm{j}}_{\mathrm{wg}}, D^{\mathrm{j}}_{\mathrm{gw}} < 0 \tag{13.81}$$

and

$$D^{\mathrm{j}}_{\mathrm{ww}} D^{\mathrm{j}}_{\mathrm{gg}} > D^{\mathrm{j}}_{\mathrm{wg}} D^{\mathrm{j}}_{\mathrm{gw}} \tag{13.82}$$

in $\mathcal{D}$. Because we are modifying only the transition model, conditions (13.37) and (13.42) remain unchanged.

We first consider the solvability of $(\mathrm{Q}^{\mathrm{l}})$. As in the base case the behavior of solutions depends critically on the location of the equilibrium point $(S^-_{\mathrm{w}}, S^-_{\mathrm{g}})$. Inequalities (13.81) and (13.82) imply that the diffusion matrix is positive definite. This means that the number and location of equilibrium points in $(\mathrm{Q}^{\mathrm{l}})$ and $(\mathrm{P}^{\mathrm{l}})$ are identical. Of course the curves where $\mathrm{d}S_{\mathrm{w}}/\mathrm{d}\xi = 0$ and $\mathrm{d}S_{\mathrm{g}}/\mathrm{d}\xi = 0$ are different. Two typical cases are shown in Figure 13.11, where we introduced again the variable $\zeta = -\xi$ (i.e. we computed orbits in the positive ζ direction).
As in the base case, equilibrium points $(S^-_{\mathrm{w}}, S^-_{\mathrm{g}}) \in \mathcal{D}_l$ (bottom figure) cannot be reached. What remains is again the possibility $(S^-_{\mathrm{w}}, S^-_{\mathrm{g}}) \in l$. Selecting points on the curve l, corresponding initial points $S^{\mathrm{l}}_{\mathrm{w}}$ were found numerically yielding a dependence which closely resembles the one shown in Figure 13.7. Observe from Figure 13.11 that now the water saturation in the transition region is not monotone: in the direction of negative ξ it first increases, reaches a global maximum and then decreases towards S^-_{w} at $\xi = -\infty$.

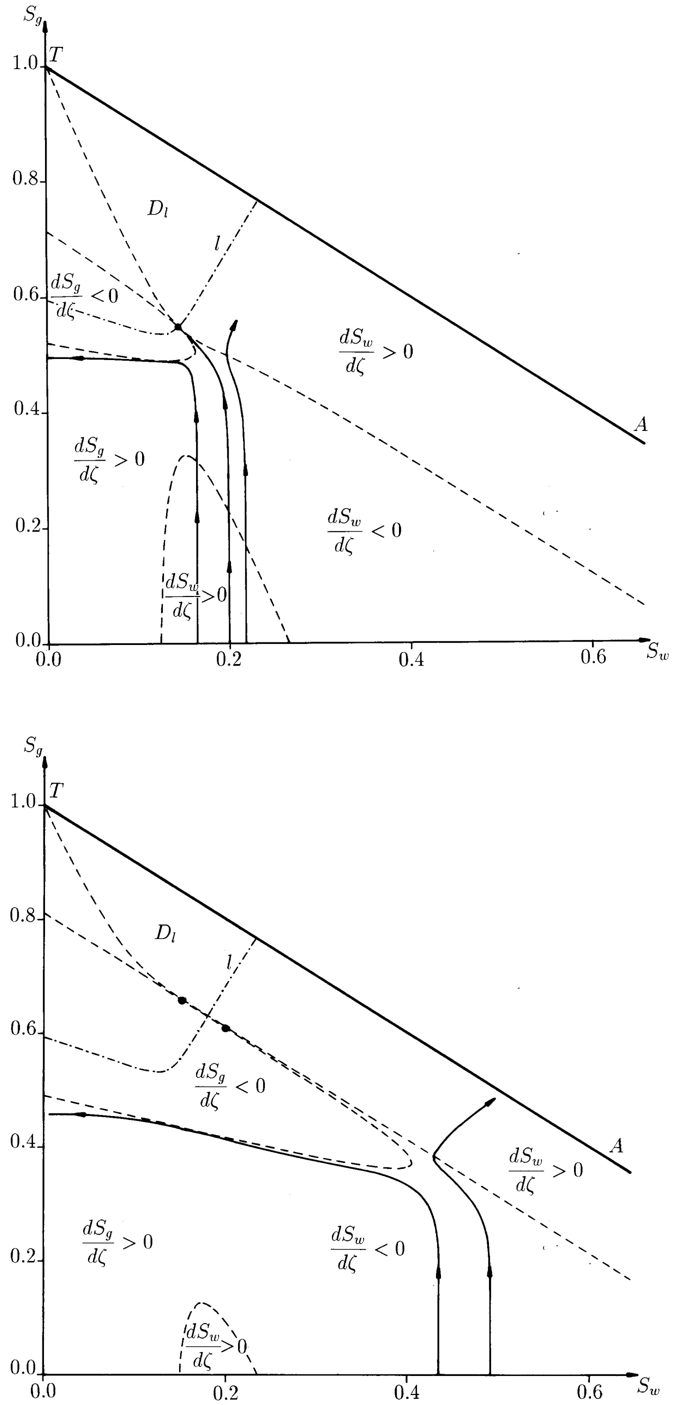


Figure 13.11. *Shooting procedure to solve* $(\mathrm{Q}^{\mathrm{l}})$. *Here* $S_{\mathrm{g}}(0) = 0$ *Top: flow diagram for* $(S_{\mathrm{w}}^-, S_{\mathrm{g}}^-) \in l$. *Bottom: flow diagram for* $(S_{\mathrm{w}}^-, S_{\mathrm{g}}^-) \in \mathcal{D}_l$. *The dots indicate the location of equilibrium points. Again* $\zeta = -\xi$

We established computationally that solutions of $(\mathrm{Q}^{\mathrm{r}})$ satisfy $\mathrm{d}S_{\mathrm{g}}/\mathrm{d}\zeta > 0$ for S_{g} close to zero. Together with the boundary conditions this implies $S_{\mathrm{g}}(\xi) = 0$ for all $\xi \geqslant 0$. A similar argument as in Section 13.3.3 gives here again $S_{\mathrm{w}}(\xi) = S_{\mathrm{w}}^{+}$ for all $\xi \geqslant 0$. We then apply the procedure outlined in Section 13.3.2 and find for different values of the sorting factor λ_{s}, as appearing in expressions (13.12), different interface saturations. Corresponding to $\lambda_{\mathrm{s}} = 2$ there results:

$$S_{\mathrm{w}}^{-} = 0.1452 \ , \quad S_{\mathrm{g}}^{-} = 0.5467 \ , \quad S_{\mathrm{w}}^{+} = 0.1990 \ , \quad \Lambda = 0.9855 \ . \tag{13.83}$$

13.4.2 Temperature variation

Next we modify the temperature distribution in the transition model. Instead of the discontinuous temperature (13.8), we will now investigate the consequence of the continuous expression (13.14). Clearly this modification leaves the transition model for $\xi < 0$ unchanged. In particular conditions (13.64) and (13.42), the Rankine–Hugoniot conditions (13.60) and the results for problem $(\mathrm{P}^{\mathrm{l}})$, with $S_{\mathrm{g}}(0) = 0$, are the same as in the base case. Thus with reference to Figure 13.7, we use the same $S_{\mathrm{w}}^{\mathrm{l}}$ curve.

The only change occurs in $(\mathrm{P}^{\mathrm{r}})$ where now the temperature variation with ξ enters in the fractional flow functions $(f_{\mathrm{i}}^{\mathrm{r}} = f_{\mathrm{i}}(S_{\mathrm{w}}, S_{\mathrm{g}}, T(\xi))$ through the mobility ratios. This dependence has no consequence for the steam saturation downstream the SCF. Since $u^{+} f_{\mathrm{g}}^{r} - vS_{\mathrm{g}} < 0$ for small positive values of S_{g}, the only possible solution satisfying the S_{g}- equation and boundary conditions is $S_{\mathrm{g}}(\xi)$=0 for all $\xi \geqslant 0$. What remains is the S_{w}- equation

$$\frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}\xi} = u^{+} f_{\mathrm{w}}(S_{\mathrm{w}}, T(\xi)) - vS_{\mathrm{w}} - (u^{+} f_{\mathrm{w}}^{+} - vS_{\mathrm{w}}^{+}) \tag{13.84}$$

for $\xi > 0$. Using the exponential relation in (13.14), we write this equation with the temperature as independent variable

$$\frac{\mathrm{d}S_{\mathrm{w}}}{\mathrm{d}T} = \frac{u^{+} f_{\mathrm{w}}(S_{\mathrm{w}}, T) - vS_{\mathrm{w}} - (u^{+} f_{\mathrm{w}}^{+} - vS_{\mathrm{w}}^{+})}{-\alpha(T - T_{\mathrm{o}})} \tag{13.85}$$

with $T_{\mathrm{o}} < T < T_1$. The corresponding boundary conditions are

$$S_{\mathrm{w}}(T_{\mathrm{o}}) = S_{\mathrm{w}}^{+} \quad \text{and} \quad S_{\mathrm{w}}(T_1) = S_{\mathrm{w}}^{\mathrm{r}} \ . \tag{13.86}$$

Because $(T_{\mathrm{o}}, S_{\mathrm{w}}^{+})$ is a singular point of equation (13.85), we solve it backwards in T. Thus given a value for S_{w}^{+}, we start at $T = T_1$ and use the iterative shooting method again to obtain an accurate approximation to the corresponding values for $S_{\mathrm{w}}^{\mathrm{r}}$.

In particular we find for any given S_{w}^{-}, which yields a unique S_{w}^{+} from Figure 13.8, a unique water saturation at the right side of the SCF. This saturation, which is denoted by $S_{\mathrm{w}}^{\mathrm{r}}(2)$ in Figure 13.7, depends also monotonically on S_{w}^{-}. Consequently there is again exactly one intersection point at S_{w}^{-} = $S_{\mathrm{w}}^{-}(2)$. As before the values for S_{g}^{-}, S_{w}^{+} and Λ are found from (13.64), Figure 13.7 and (13.60):

$$S_{\mathrm{w}}^{-} = S_{\mathrm{w}}^{-}(2) = 0.1288 \ , \quad S_{\mathrm{g}}^{-} = 0.5337 \ , \quad S_{\mathrm{w}}^{+} = 0.2010 \ , \quad \Lambda = 0.9856 \ . \tag{13.87}$$

The composite solution as a path in the saturation-temperature space is shown as curve 2 in Figure 13.9. Note the significant change in the transition region, in particular the striking non-monotonicity of S_{w}, but the minor change in the hyperbolic part of the path, i.e. the outer solution.

13.4.3 Positive steam saturation at SCF

Finally we modify the base case by replacing condition (13.67). Now we assign a positive value $S_{\mathrm{g}}(0)$ to the steam saturation at the SCF. This does not involve conditions (13.64), (13.42) and (13.60), which therefore remain unchanged here. To find the saturations in the transition region, we now have to solve subproblems $(\mathrm{P}^{\mathrm{l}})$ and $(\mathrm{P}^{\mathrm{r}})$ subject to $S_{\mathrm{g}}^{l} = S_{\mathrm{r}}^{r} = S_{\mathrm{g}}(0) > 0$. With reference to Figure 13.12, we apply iterative shooting procedures starting from the line $S_{\mathrm{g}} = S_{\mathrm{g}}(0)$: problem $(\mathrm{P}^{\mathrm{l}})$ is solved backwards in ξ (or as before, in positive $\zeta = -\xi$ direction) and $(\mathrm{P}^{\mathrm{r}})$ is solved forwards in ξ.

Given S_{w}^{-}, we first determine S_{w}^{+} from Figure 13.8 and then solve $(\mathrm{P}^{\mathrm{l}})$ and $(\mathrm{P}^{\mathrm{r}})$ repeatedly to obtain accurate approximations for $S_{\mathrm{w}}^{\mathrm{l}}$ and $S_{\mathrm{w}}^{\mathrm{r}}$. Again this leads to two monotone curves: $S_{\mathrm{w}}^{\mathrm{l}}$ is increasing and $S_{\mathrm{w}}^{\mathrm{r}}$ is decreasing with respect to S_{w}^{-}. The unique intersection point gives the required value for S_{w}^{-}. The saturations S_{g}^{-} and S_{w}^{+}, and the condensation rate Λ follow as before. Corresponding to $S_{\mathrm{g}}(0) = 0.035$, the result is:

$$S_{\mathrm{w}}^{-} = 0.1237 \ , \quad S_{\mathrm{g}}^{-} = 0.5339 \ , \quad S_{\mathrm{w}}^{+} = 0.2015 \ , \quad \Lambda = 0.9856 \ . \tag{13.88}$$

Given the parameter values in Table II, one cannot obtain a solution for significant larger $S_{\mathrm{g}}(0)$ values. This follows from the sign of the right side of the gas equation in $(\mathrm{P}^{\mathrm{r}})$. Taking $S_{\mathrm{w}} = 0.2$ in $u^{+} f_{\mathrm{g}}^{\mathrm{r}} - vS_{\mathrm{g}}$, one finds that this expression is negative for $0 < S_{\mathrm{g}} < 0.04$ and positive for larger S_{g} values. Hence, only when $S_{\mathrm{g}}(0)$ is taken in this range a decreasing gas saturation can be constructed. This limitation is a direct consequence of the large viscosity ratio M_{og}

13.5 Parameter variation

Besides the mathematical context, the solutions constructed in Sections 13.3 and 13.4 are of interest to petroleum engineers, because they can be used to

1. interpret one dimensional tube experiments;
2. validate thermal simulators for steamdrive;
3. quantify the influence of reservoir and rock properties.

In this section we focus on the parameter dependence, which we investigate for the average oil saturation in the steam zone $\bar{S}_{\mathrm{o}}$. Integrating the mass balance for oil, this quantity can be expressed directly in terms of the upstream saturations at the SCF. These saturations result directly from our analysis. As in DAKE [19] and DULLIEN [20] we find

$$\bar{S}_{\mathrm{o}} = S_{\mathrm{o}}^{-} - \frac{u}{v} f_{\mathrm{o}}(S_{\mathrm{w}}^{-}, S_{\mathrm{g}}^{-}) \ . \tag{13.89}$$

Now given a set of model parameters, we compute S_{w}^{-}, S_{g}^{-} and S_{o}^{-} as explained in this chapter. This analysis, however, is rather involved and it would be desirable to find the upstream saturations by means of a relatively straightforward approximation. With reference to Figures 13.6, 13.9, 13.11 and 13.12 it seems natural to choose the minimum of the curve l for that purpose. Denoting this point by $(S_{\mathrm{w}}^{\min}, S_{\mathrm{g}}^{\min}, S_{\mathrm{o}}^{\min})$ we then have that $S_{\mathrm{w}}^{\min}$ and $S_{\mathrm{g}}^{\min}$ satisfy (13.64) for the smallest possible gas saturation. Using these values in (13.89) yields $\bar{S}_{\mathrm{o}}^{\min}$ as the approximate average oil saturation in the steam zone.

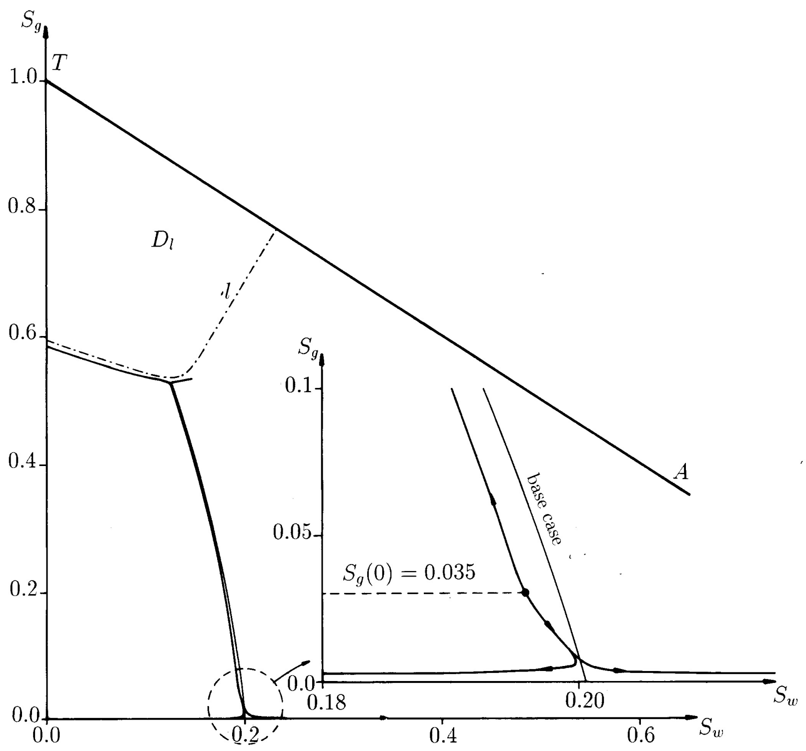


Figure 13.12. *Orbits in the saturation space, starting with a positive value of the steam saturation at the* SCF, $S_g(0) > 0$

In the parameter variation we change each time only one parameter in Table I. Instead of Table II, we will use here three phase permeabilities. These are obtained by combining Corey two phase relative permeabilities and the modified Stone I method, see FAYERS & MATTHEWS [23]. In full dimensional form they read

$$\begin{aligned}
k_{\mathrm{rw}} = k_{\mathrm{rw}}(S_{\mathrm{w}}) \quad &= k'_{\mathrm{rw}} S_{\mathrm{we}}^{\frac{2+3\lambda_{\mathrm{s}}}{\lambda_{\mathrm{s}}}} \ , \\
k_{\mathrm{rg}} = k_{\mathrm{rg}}(S_{\mathrm{g}}) \quad &= k'_{\mathrm{rg}} S_{\mathrm{ge}}^2 \left(1 - (1 - S_{\mathrm{ge}})^{\frac{2+\lambda_{\mathrm{s}}}{\lambda_{\mathrm{s}}}}\right) , \\
k_{\mathrm{ro}} = k_{\mathrm{ro}}(S_{\mathrm{w}}, S_{\mathrm{g}}) &= \frac{S_{\mathrm{o}}}{k_{\mathrm{rcow}}(1 - S_{\mathrm{w}})(1 - S_{\mathrm{ge}})} k_{\mathrm{row}} k_{\mathrm{rog}} \ .
\end{aligned}$$

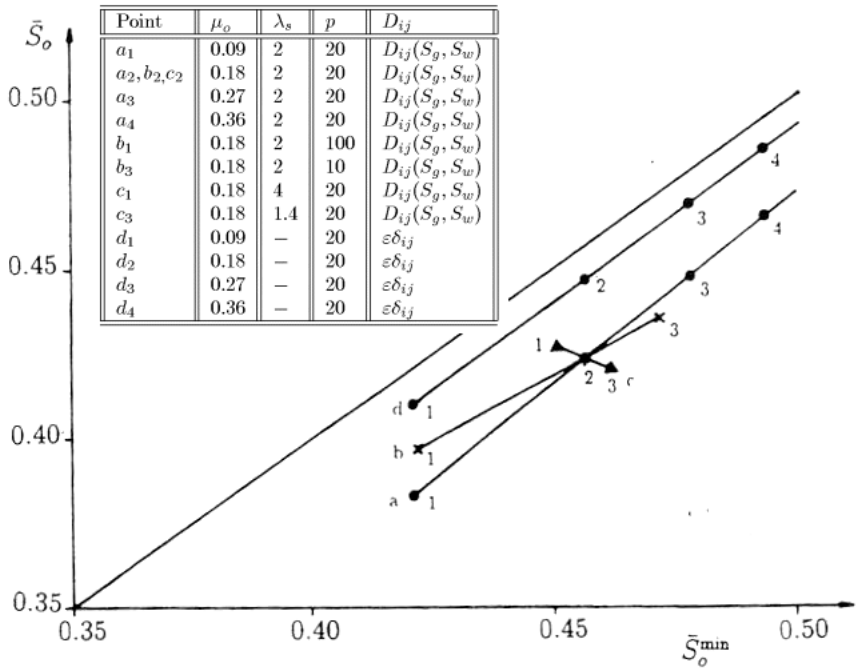
Here

$$S_{\mathrm{we}} = \frac{S_{\mathrm{w}} - S_{\mathrm{wc}}}{1 - S_{\mathrm{wc}}} \ , \ S_{\mathrm{ge}} = \frac{S_{\mathrm{g}}}{1 - S_{\mathrm{wc}}}$$

and

$$k_{\mathrm{row}} = k'_{\mathrm{rg}} (1 - S_{\mathrm{we}})^2 \left(1 - S_{\mathrm{we}}^{\frac{2+\lambda_{\mathrm{s}}}{\lambda_{\mathrm{s}}}}\right) \ , \ k_{\mathrm{rog}} = k'_{\mathrm{rw}} (1 - S_{\mathrm{ge}})^{\frac{2+3\lambda_{\mathrm{s}}}{\lambda_{\mathrm{s}}}} \ .$$

We use k'_{rw}=0.5 for the end-point permeability of the wetting phase at residual non-wetting phase saturation and k'_{rg}=1.0 for the end-point permeability of the non-wetting phase at connate wetting phase saturation. Finally we set $k_{\mathrm{rcow}} = 1$. To describe the effect of oil film flow (oil may spread on water in the presence of steam), the expressions for k_{row} and k_{rog} are different from the ones proposed by FAYERS & MATTHEWS [23].



Point	μ_o	λ_s	p	D_{ij}
a_1	0.09	2	20	$D_{ij}(S_g, S_w)$
a_2, b_2, c_2	0.18	2	20	$D_{ij}(S_g, S_w)$
a_3	0.27	2	20	$D_{ij}(S_g, S_w)$
a_4	0.36	2	20	$D_{ij}(S_g, S_w)$
b_1	0.18	2	100	$D_{ij}(S_g, S_w)$
b_3	0.18	2	10	$D_{ij}(S_g, S_w)$
c_1	0.18	4	20	$D_{ij}(S_g, S_w)$
c_3	0.18	1.4	20	$D_{ij}(S_g, S_w)$
d_1	0.09	–	20	$\varepsilon\delta_{ij}$
d_2	0.18	–	20	$\varepsilon\delta_{ij}$
d_3	0.27	–	20	$\varepsilon\delta_{ij}$
d_4	0.36	–	20	$\varepsilon\delta_{ij}$

Figure 13.13. *Comparison of average oil saturation calculated from full computations and calculated with the "approximate minimum condition"*

Obviously these permeabilities change the nature of the Jacobian matrix (13.29), and in particular of its eigenvalues. They may become complex, yielding an elliptic region in the saturation triangle, see GUZMÁN & FAYERS [32]. However for our parameter choice, i.e. the large M_{og}, the small elliptic region is situated near the $S_{\mathrm{g}} = 0$ axis and plays no role in the analysis.

We show the computational results in Figure 13.13. The vertical axis gives $\bar{S}_{\mathrm{o}}$, as established with the procedure outlined in Sections 13.3 and 13.4. The horizontal axis shows the "minimum"approximation $\bar{S}_{\mathrm{o}}^{\min}$. Line d in Figure 13.13 shows results for various cold oil viscosities in the medium viscosity range, i.e. between 0.09-0.36 [Pa s], using a saturation independent capillary diffusion. Observe that the result is nearly parallel and fairly close to the $\bar{S}_{\mathrm{o}} = \bar{S}_{\mathrm{o}}^{\min}$ line. As to be expected, an increasing oil viscosity leads to a deteriorating displacement efficiency with an increasing oil saturation in the steam zone. In all other cases shown in Figure 13.13, we use Brooks–Corey capillary diffusion. Along line a we vary again the viscosity as for line d. Note the significant differences caused by the different capillary behavior in the transition zone. Along line c we vary the sorting factor λ_{s}. This affects both the relative permeabilities and the capillary diffusion. Observe that the deviation from the $\bar{S}_{\mathrm{o}} = \bar{S}_{\mathrm{o}}^{\min}$ line decreases with λ_{s}. For reasons of practical interest we also show the effect of different pressures. The steam pressure is not explicit in our equations but affects a number of parameters in Table I. We use empirical relations given by TORTIKE & FAROUQ ALI [71] to represent the steam tables. The pressure clearly determines the (boiling) temperature. It also has a small effect on the enthalpy for the conversion of cold water to hot steam (ΔH). Therefore the steam condensation front velocity decreases at higher pressures, see (13.4). Through its influence on temperature, a high pressure enhances the steam viscosity and lowers the liquid viscosities. Direct pressure effects on viscosities are negligible. The pressure range is between 10 and 100 bar. Indeed the displacement efficiency improves with increasing pressure. Note that this occurs at the expense of a much higher mass of injected steam per unit volume of recovered oil, because higher temperatures are involved now; the reservoir must be heated to a higher temperature.

13.6 Conclusions

Based on the results of this chapter we conclude the following:

- The steamdrive model considered in this chapter gives a transitional shock wave at the steam condensation front.
- As a consequence, the shock conditions at the steam condensation front inherit details of the local parabolic transition model.
- The presence of steam in the downstream part of the transition zone has no significant effect on the results.
- The rate of temperature decline has no significant effect outside the transition zone, i.e. in the hyperbolic limit.
- The effect of Brooks–Corey capillary diffusion instead of constant (saturation independent) capillary diffusion is well noticeable and cannot be disregarded.
- An approximate solution is given, based on the minimum of the l-curve in domain $\mathcal{D}$. The validity of this approximation can be checked from Figure 13.13 for different values of the model parameters.
- The water saturation is significantly non-monotone when considering a continuous temperature decline in the transition region (Figure 13.9). The maximum does not depend on the small parameter ε and persists in the hyperbolic limit.

14 Exercises

1. *Consider the equation*

$$u_t + (f(u))_x = \nu u_{xx} \qquad \textit{in } Q\ ,$$

where $f \in C^2(\mathbb{R})$, $f'' > 0$ *and* $\nu > 0$. *Show that a travelling wave solution* $u(x,t) = w(\eta)$, *with* $\eta = x - ct$ *and* $w(-\infty) = u_\mathrm{l}$, $w(+\infty) = u_\mathrm{r}$, *exists if and only if* $u_\mathrm{l} > u_\mathrm{r}$.

2. *Consider the nonlinear convection-diffusion equation*

$$\begin{cases} u_t + \big(f(u)\big)_x = \nu\big(D(u)u_x\big)_x \\ 0 \leqslant u \leqslant 1 \end{cases} \qquad \textit{in } Q\ ,$$

where $f : [0,1] \to \mathbb{R}$ *is smooth and strictly convex,* $\nu > 0$ *and where* $D : [0,1] \to [0,\infty)$ *is given by*

$$D(u) = u^\alpha (1-u)^\beta \qquad \textit{for } 0 \leqslant u \leqslant 1\ ,$$

with $\alpha, \beta > 0$.

(i) Show that a travelling wave exists if and only if $0 \leqslant u_\mathrm{r} \leqslant u_\mathrm{l} \leqslant 1$;

(ii) Investigate the wave profile for the cases:

- $0 < u_\mathrm{r} < u_\mathrm{l} < 1$;
- $0 = u_\mathrm{r} < u_\mathrm{l} < 1$;
- $0 < u_\mathrm{r} < u_\mathrm{l} = 1$;
- $0 = u_\mathrm{r} < u_\mathrm{l} = 1$.

3. *The single hump solution for the Burgers equation (Section 1.2) results in the problem*

$$\begin{cases} -\eta\varphi + \varphi^2 = 2\nu\varphi' + A \quad \textit{for } -\infty < \eta < \infty\ , \\ \lim\limits_{|\eta|\to\infty} \eta\,\varphi(\eta) = 0\ , \quad \displaystyle\int_{\mathbb{R}} \varphi(\eta)\mathrm{d}\eta = M\ , \end{cases}$$

where A *is a constant of integration. Show that* $A = 0$.

4. *Derive expression* (1.11).

5. *Proof Proposition 2.17.*

6. *Averaging gravity induced fingers in porous media flow results in the problem*

$$\begin{cases} \dfrac{\partial\rho}{\partial t} - \Gamma\dfrac{\partial}{\partial z}\rho(1-\rho) = 0 \quad \textit{for} \ -a < z < a, \ t > 0 \ , \\[2ex] \rho(z,0) = \begin{cases} 1 & \textit{as}\ 0 < z < a \ ; \\ 0 & \textit{as}\ -a < z < 0 \ , \end{cases} \\[2ex] \rho(1-\rho) = 0 \quad \textit{at}\ z = \pm a, \ t > 0 \ \textit{(zero flux)} \ . \end{cases}$$

Here ρ *denotes the averaged fluid density and* Γ, a *are positive constants. Determine the solution by the method of characteristics. Show that a conversion time* $T > 0$ *exists such that*

$$\rho(z,t) = \begin{cases} 0 & \textit{as}\ 0 < z < a \ , \ t > T \ , \\ 1 & \textit{as}\ -a < z < 0 \ , \ t > T \ . \end{cases}$$

7. *The transport of a reactive solute in a porous column, undergoing non-equilibrium adsorption, is described by the coupled system (e.g.* VAN DUIJN & KNABNER [76, 77]*)*

$$\frac{\partial}{\partial t}(u+v) + q\frac{\partial u}{\partial x} = 0 \ , \tag{14.1a}$$

$$\frac{\partial v}{\partial t} = k\{\varphi(u) - v\} \ , \tag{14.1b}$$

where $-\infty < x < \infty$ *and* $t > 0$*. Here* u *denotes the concentration of the chemical species in the fluid and* v *the concentration adsorbed on the porous matrix. Further,* $q > 0$ *is the averaged fluid velocity and* $k > 0$ *the reaction rate constant. Finally,* $\varphi : [0,\infty) \to [0,\infty)$ *is the adsorption isotherm satisfying*

$$\begin{aligned} &\varphi \in C^\infty((0,\infty)) \cap C([0,\infty)) \ , \\ &\varphi(0) = 0 \ , \\ &\varphi'(s) > 0 \ , \ \varphi''(s) < 0 \quad \textit{for}\ s > 0 \ . \end{aligned}$$

(i) Show that travelling wave solutions exist satisfying

$$\begin{aligned} u(-\infty,t) &= u_l > 0 \ , & u(+\infty,t) &= 0 \ , \\ v(-\infty,t) &= v_l := \varphi(u_l) \ , & v(+\infty,t) &= 0 \ , \end{aligned}$$

for all $t > 0$*;*

(ii) Compute the travelling waves for the Freundlich isotherm, when $\varphi(u) = u^p$, $0 < p < 1$*;*

(iii) Study the limit $k \to \infty$ *(equilibrium adsorption);*

(iv) Are travelling waves possible when $v_l \neq \varphi(u_l)$ *?*

8. *Letting* $k \to \infty$ *in* (14.1a), (14.1b) *results in the reduced problem*

$$\frac{\partial}{\partial t}(u + \varphi(u)) + q\frac{\partial u}{\partial x} = 0 \quad \textit{for} \ -\infty < x < \infty \ , \ t > 0 \ , \tag{14.2}$$

describing the transport of a solute undergoing fast or equilibrium adsorption.

(i) *Determine the solution of the Riemann problem with*

$$u(x,0) = \begin{cases} u_\mathrm{l} \geqslant 0 & \text{as } x < 0 \;, \\ u_\mathrm{r} \geqslant 0 & \text{as } x > 0 \;; \end{cases}$$

(ii) *What is wrong in the following steps:*

$$(14.2) \Rightarrow \big(1+\varphi'(u)\big)\frac{\partial u}{\partial t} + q\frac{\partial u}{\partial x} = 0 \Rightarrow$$
$$\Rightarrow \frac{\partial u}{\partial t} + \frac{q}{1+\varphi'(u)}\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial t} + q\frac{\partial f(u)}{\partial x} = 0 \;, \quad (14.3)$$

with $f(u) = \int_0^u \frac{1}{1+\varphi'(s)}\mathrm{d}s$. *Compare the solutions of the Riemann problems* ($u_\mathrm{l} > u_\mathrm{r}$) *for* (14.2) *and* (14.3).

9. Buckley–Leverett with gravity

(i) *Include gravity in the two-phase water-oil flow as presented in the Appendix. Show that the water saturation now satisfies*

$$\Phi\frac{\partial S_\mathrm{w}}{\partial t} + \frac{\partial}{\partial z}\left\{ \frac{\frac{k_\mathrm{w}}{\mu_\mathrm{w}}\frac{k_\mathrm{o}}{\mu_\mathrm{o}}}{\frac{k_\mathrm{w}}{\mu_\mathrm{w}} + \frac{k_\mathrm{o}}{\mu_\mathrm{o}}} q - (\gamma_\mathrm{w} - \gamma_\mathrm{o})\frac{\frac{k_\mathrm{w}}{\mu_\mathrm{w}}\frac{k_\mathrm{o}}{\mu_\mathrm{o}}}{\frac{k_\mathrm{w}}{\mu_\mathrm{w}} + \frac{k_\mathrm{o}}{\mu_\mathrm{w}}} \right\} = 0 \;, \quad (14.4)$$

Here z *denotes the vertical coordinate, pointing upwards against the direction of gravity.*

(ii) *Setting*

$$S := \frac{S_\mathrm{w} - S_\mathrm{wc}}{1 - S_\mathrm{wc} - S_\mathrm{w}} \;, \quad k_\mathrm{w} = kS^2 \;, \quad k_\mathrm{o} = k(1-S)^2 \;,$$

show that (14.4) *can be put in the dimensionless form*

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial z}\{f_\mathrm{w}(S) - N_\mathrm{g}H(S)\} = 0 \;, \quad (14.5)$$

where

$$f_\mathrm{w}(S) = \frac{MS^2}{(1-S)^2 + MS^2} \;, \quad M = \frac{\mu_\mathrm{o}}{\mu_\mathrm{w}} \;;$$
$$H(S) = (1-S)^2 f_\mathrm{w}(S) \;;$$
$$N_\mathrm{g} = \frac{(\gamma_\mathrm{w} - \gamma_\mathrm{o})k}{q\mu_\mathrm{w}} \quad \textit{(gravity number)} \;.$$

Note that the flux

$$F(S) := f_\mathrm{w}(S) - N_\mathrm{g}H(S)$$

is non-monotone for $N_\mathrm{g} > 1$.

(iii) Let $N_{\mathrm{g}} > 1$. *Solve the Riemann problem for* (14.5) *with*

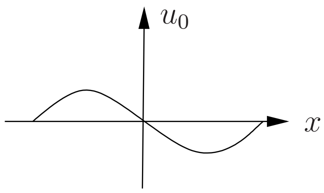
$$S(z,0) = \begin{cases} 1 & \text{as } z < 0 \\ 0 & \text{as } z > 0 \end{cases} \quad \text{and} \quad S(z,0) = \begin{cases} 0 & \text{as } z < 0\,, \\ 1 & \text{as } z > 0\,. \end{cases}$$

10. *Consider Proposition 3.3 and Corollary 3.4. Suppose* $s^+(t)$ *is differentiable for some* $t_0 > T$ *(waiting time). Show that* $\dot{s}^+(t_0) > 0$.

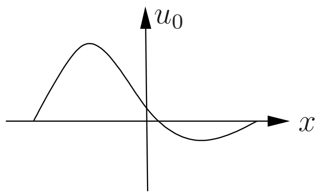
11. *Determine the large time behaviour of the unique entropy solution of the initial value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0 & \text{in } Q\,, \\ u(\cdot,0) = u_0(\cdot) & \text{on } \mathrm{I\!R}\,, \end{cases}$$

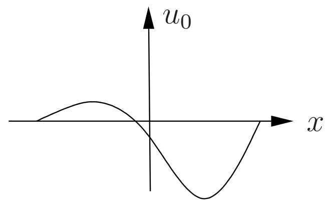
where $f \in C^2(\mathrm{I\!R})$, $f'' > 0$ *on* $\mathrm{I\!R}$ *and* $f''(0) = k > 0$. *Consider the cases:*



$$\int_{\mathrm{I\!R}} u_0(x)\mathrm{d}x = 0$$



$$\int_{\mathrm{I\!R}} u_0(x)\mathrm{d}x > 0$$



$$\int_{\mathrm{I\!R}} u_0(x)\mathrm{d}x < 0$$

12. *Find the entropy solution of the Riemann problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(a(x)\dfrac{u^2}{2}\right) = 0 & \text{in } Q\,, \\ u(x,0) = \begin{cases} u_{\mathrm{l}} & \text{as } x < 0\,, \\ u_{\mathrm{r}} & \text{as } x > 0\,, \end{cases} \end{cases}$$

where

$$a(x) = \begin{cases} a_{\mathrm{l}} & \text{as } x < 0\,, \\ a_{\mathrm{r}} & \text{as } x > 0\,. \end{cases}$$

13. *Let* $u_\varepsilon \in C^\infty(\overline{Q}) \cap L^\infty(Q)$ *satisfy*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} & \text{in } Q\,, \\ \dfrac{\partial u}{\partial x}(x,\cdot) \to 0 & \text{in } C_{\mathrm{loc}}([0,\infty)) \text{ as } |x| \to \infty\,, \end{cases}$$

where $f'' \geqslant \mu$ *in* $\mathrm{I\!R}$. *Show that*

$$\frac{\partial u_\varepsilon}{\partial x} \leqslant \frac{1}{\mu t} \quad \text{in } Q \quad \text{(entropy inequality)}\,.$$

Hint: Consider the equation for $v_\varepsilon := \dfrac{\partial u_\varepsilon}{\partial x}$ *and construct a supersolution.*

14. *Transform* (7.20) *into the standard form* (7.21).

15. *Show that the unique solution of the Riemann problem* (9.1) *must be of the form* $\mathbf{u}(x,t)=\mathbf{u}(x/t)$ *in* Q.

16. *Consider the shallow water equations* (7.10) *and let*

$$(z,v)(x,0)=\begin{cases}(z_\mathrm{l},v_\mathrm{l}) & \text{as } x<0\ ,\\ (z_\mathrm{r},v_\mathrm{r}) & \text{as } x>0\ .\end{cases}$$

Find the entropy solutions for the cases

(i) $z_\mathrm{l}=z_\mathrm{r}$, $v_\mathrm{l}>v_\mathrm{r}$;

(ii) $z_\mathrm{l}>z_\mathrm{r}$, $v_\mathrm{l}=v_\mathrm{r}$;

(iii) $z_\mathrm{l}=z_\mathrm{r}$, $-v_\mathrm{l}=v_\mathrm{r}>0$.

17. *Three phase flow of oil, water and gas in a porous medium results in the system*

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial f(u,v)}{\partial x}=0\\[2ex] \dfrac{\partial v}{\partial t}+\dfrac{\partial g(u,v)}{\partial x}=0\end{cases}\qquad \text{in } Q\ ,$$

with

$$\begin{aligned} f(u,v)&=\frac{M_\mathrm{ow}u^2}{M_\mathrm{ow}u^2+(1-u-v)^2+M_\mathrm{og}v^2}\ ,\\ g(u,v)&=\frac{M_\mathrm{og}v^2}{M_\mathrm{ow}u^2+(1-u-v)^2+M_\mathrm{og}v^2}\ ,\end{aligned}$$

$$M_\mathrm{ow},\ M_\mathrm{og}>0\quad \text{(mobility ratios)}\ .$$

Here $u\geqslant 0$, $v\geqslant 0$ *denote, respectively, the water, gas saturations, satisfying* $u+v\leqslant 1$. *Hence solutions are restricted to the saturation triangle* $\mathcal{D}:=\{(u,v): u,v\geqslant 0 \text{ and } u+v\leqslant 1\}$.

(i) Show that the eigenvalues λ_1, λ_2 *are real in* $\mathcal{D}$.

(ii) Show that $\lambda_1<\lambda_2$ *in* $\mathcal{D}$, *except at the three vertices of* $\mathcal{D}$ *and at one interior point.*

Remark 14.1. *Points where* $\lambda_1=\lambda_2$ *are called umbilic points. They may result in a disconnected Hugoniot locus as shown by* ISAACSON ET AL. [41].

18. *Consider the simplified three phase system*

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial f(u,v)}{\partial x}=0\ ,\\[2ex] \dfrac{\partial v}{\partial t}+\dfrac{\partial g(v)}{\partial x}=0\ ,\end{cases}\qquad \text{in } Q\ ,$$

with

$$f(u,v) = \frac{u}{1+(\alpha-1)v} \quad \textit{and} \quad g(v) = \frac{\alpha v}{1+(\alpha-1)v} \ ,$$

and $\alpha > 1$. *Follow the steps of Chapter 9 to solve the Riemann problem (in* $\mathcal{D}$*) with:*

$$(u,v)(x,0) = \begin{cases} (u_\mathrm{l}, v_\mathrm{l}) & \textit{as } x < 0 \ , \\ (u_\mathrm{r}, v_\mathrm{r}) & \textit{as } x > 0 \ . \end{cases}$$

In other words:

(i) Determine the eigenvalues and eigenvectors of the associated Jacobian matrix.

(ii) Determine the Hugoniot locus for a typical state in $\mathcal{D}$.

(iii) Which shocks are admissible ?

(iv) Determine the rarefactions in $\mathcal{D}$.

(v) Complete the construction of an admissible solution.

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